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ELECTROMAGNETIC DIFFRACTION BY A PERFECTLY-CONDUCTING PLANE ANGULAR SECTION

The Ohio State University

ElectroScience Laboratory

Department of Electrical Engineering
Columbus, Ohio 43212

Scientific Report No. 2
Contract Number AF19(628)-5929
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Task No. 563502
Work Unit No. 56350201

23 March 1970

Contract Monitor: John K. Schindler
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Air Force Cambridge Research Laboratories
Office of Aerospace Research
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ELECTROMAGNETIC DIFFRACTION BY A PERFECTLY-CONDUCTING PLANE ANGULAR SECTION

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FOREWORD

This report, OSURF Report Number 2183-2, was prepared by The ElectroScience Laboratory, Department of Electrical Engineering, The Ohio State University at Columbus, Ohio. Research was conducted under Contract AF 19(628)-5929. Dr. John K. Schindler, CRDG of the Air Force Cambridge Research Laboratories at Bedford, Massachusetts was the Program Monitor for this research.

ABSTRACT

In this report, the exact solution for the electromagnetic field diffracted by a perfectly conducting plane angular sector is determined. The problem is a three-dimensional vector problem and the solution is presented in the form of a dyadic Green's function, which is the most general form of solution possible. Thus the vector fields as well as the current on the sector may be determined for any given source excitation.

The corner angle of the plane angular sector is arbitrary, varying between zero and π. Special cases of the plane angular sector are the half plane and the quarter plane. To find the fields for larger angles, such as at the corner of an aperture, Babinet's principle can be used.

The dyadic Green's function is composed of vector wave functions, which in turn are composed of scalar wave functions. The problem is solved in a sphero-conal coordinate system. In this system, the plane angular sector is one of the coordinate surfaces, so that the separation of variables technique is used to find the scalar wave functions. They consist of spherical Bessel's functions and Lamé functions. The Lamé functions are solutions of two coupled differential equations. These equations are solved for the special case of a quarter plane scatterer. The first 192 eigenvalues and eigenfunctions are computed and tabulated.

The fields and currents close to the tip of the quarter plane are presented. These fields and currents have been the subject of much conjecture by several authors. It is shown that the dominant field behaves as $r^{\nu-1}$, where the lowest value of the eigenvalue ν is 0.296. The far fields for infinitesimal dipole sources very close to the tip are also determined and several patterns are presented.

As with any exact solution of a complex problem, the results are not simple. It is felt, however, that the exact solution obtained here lays the foundation for subsequent work. For instance, it should be possible to use this work to determine an asymptotic approximation and thus derive a "diffraction coefficient" for the tip. This would be very useful in Keller's "Geometrical Theory of Diffraction." Without any additional work, there are sufficient numerical results presented in this report to determine the fields within approximately one wavelength of the tip for any source, or the fields everywhere for a source within one wavelength of the tip.

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CHAPTER I

INTRODUCTION

The primary purpose of this work is to determine the exact solution for the electromagnetic field scattered by a perfectly conducting plane angular sector. A plane angular sector is the section of an infinite plane bounded by two interesting straight edges which terminate at a corner. This is a three-dimensional vector problem, so the final solution is in the form of a dyadic Green's function. The physical configuration is shown in Figure 1. The corner angle is arbitrary in the range zero to π. For larger corner angles, such as the corner of an aperture, Babinet's principle can be used to obtain the fields diffracted by the complementary structure.

Diffraction by objects with edges, corners, tips, etc., has occupied the attention of several authors.* The classical case is the half-plane, which was solved by Sommerfeld, and has since been studied by many authors.* The problem considered here is more general, the half-plane being a special case of the plane angular sector. The solution near the corner, or tip, is of primary interest in the angular sector problem. A half-plane edge is a discontinuity in the scattering surface since the normal to the surface cannot be defined. A tangent can be defined, however. The corner of a plane angular sector is a "double" discontinuity since even the tangent cannot be defined there.

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*For extensive bibliographies, see Reference 17, Ch. 12, and Reference 19.

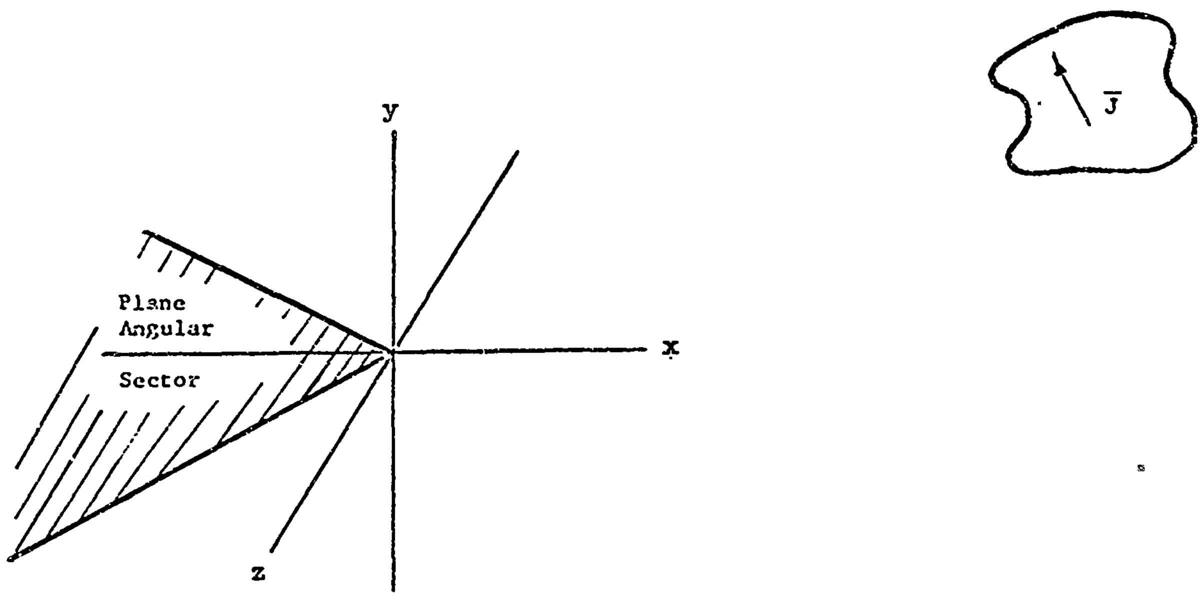

Fig. 1. Physical configuration

Several authors [1, 2, 3, 4, 5] have considered the field variation at this "double" discontinuity and have made conjectures based on approximations and physical reasoning. Two of these cases are compared with the exact solution in Chapter 6. The effect of the corner on the far field is also of interest. By using the well known asymptotic solution of the edge diffraction problem, and the reflection from a plane, it should be possible to identify a "diffraction coefficient" for the tip. This has not been done in this work; however, it is felt that the exact solution presented here is a good starting point for such an endeavor.

As far as the authors could determine, there is no published work on the electromagnetic diffraction by a plane angular sector. The scalar problem has been solved by Kraus [1], Radlow [2], and Kraus and Levine [6]. Radlow has considered the problem of the diffraction of a scalar plane wave by a quarter plane. He determined a two variable integral representation of the scattered field. Using a generalization, or extension, of the classical one variable Wiener-Hopf method, he then found the transform solution that forces the total field to zero on the quarter plane. It should be emphasized that his scalar solution is not applicable to the electromagnetic case.

Kraus has approached the problem from another point of view. In his dissertation he developed a "uniformized" sphero-conal coordinate system, and determined a scalar Green's function in this coordinate system. His scattering body is a plane angular sector, which is a degenerate case of one of the elliptic cone coordinate surfaces. An eigenfunction expansion was used to determine the Green's function for both the hard and soft boundary conditions on the sector. The angular functions in the eigenfunction expansion are called Lamé functions and are solutions of two coupled Lamé equations. The first eigenvalue of these equations for both boundary conditions was determined approximately for several sector angles.

The article by Kraus and Levine was published several years later. It contains some refinement of the problem and the general solution for an elliptic cone scattering body; no numerical results were presented.

The "uniformized" sphero-conal coordinate system is used in this paper to solve the vector problem. This is one of the six coordinate systems in which the vector wave equation is separable. The coupled Lamé equations which occur are solved in this paper for the special case of the quarter plane. The Lamé functions are a very general class of functions. They depend on an ellipticity parameter and two eigenvalues. As the ellipticity goes to one and the elliptic cones become circular cones, the Lamé functions reduce to Legendre functions and trigonometric functions. For other limiting values of the ellipticity and the eigenvalues, they become Tchebycheff polynomials and Mathieu functions.

This report has been organized in the same order as the problem was solved. In order to determine the dyadic Green's function, it is necessary to have a complete set of vector wave functions, and in order to have a complete set of vector wave functions for this problem, two complete sets of scalar wave functions are needed. Before the scalar wave functions can be determined, it is necessary to understand the coordinate system.

Consequently, this report begins with a discussion of the sphero-conal coordinate system in Chapter 2. The peculiarities of the coordinate system are discussed and the effect of these peculiarities on the scalar wave functions is discussed.

In Chapter 3 the scalar wave equation for this coordinate system is presented and then separated into the spherical Bessel's equation and the coupled Lamé equations. The boundary conditions are examined and the problem is separated into four different boundary value problems. The method for solving these problems is discussed and then the first 192 eigenvalues and eigenfunctions are determined for the quarter plane.

The vector wave functions are determined in Chapter 4, and in Chapter 5 the dyadic Green's function is derived. Since the vector wave functions are a new set of functions, it is necessary to investigate them in some detail in order to determine the dyadic Green's function. This investigation is chiefly concerned with orthogonality properties and normalization. A suggestion concerning normalization of the Lamé functions is also included in Chapter 5.

Some numerical results for the fields diffracted by and the currents on a quarter plane are presented in Chapter 6. The dominant behavior of the fields and current near the tip and along the edges is examined. This is done for both an infinitesimal dipole and a plane wave source. The far field due to an infinitesimal dipole very close to the tip is also discussed and some patterns are presented.

The appendices are all connected with Chapter 3. Appendix A includes a discussion of some of the self-adjoint properties of the scalar wave equation. Appendices B and C describe the calculation of the eigenvalues and eigenfunctions in detail, and Appendix D describes another approach for calculating the eigenvalues and eigenfunctions.

CHAPTER II

COORDINATE SYSTEM

This chapter contains a description of the coordinate system and its peculiarities, and the effect of these peculiarities on solutions of the wave equation.

Description of the Coordinate System

The uniformized sphero-conal coordinate system (r,θ,φ) [1,6] is introduced by the following coordinate transformation:

$$x = r \cos \theta \sqrt{1-k'^2 \cos^2 \varphi} \tag{2.1a}$$

$$y = r \sin \theta \sin \varphi \tag{2.1b}$$

$$z = r \cos \varphi \sqrt{1-k^2 \cos^2 \theta} \tag{2.1c}$$

where

$$k'^2 = 1-k^2, \quad 0 \leq k^2 \leq 1$$

$$0 \leq \theta \leq \Pi$$

$$0 \leq \varphi \leq 2\Pi$$

$$r \geq 0$$

and x, y, and z are the usual Cartesian coordinates.

It is derived from the standard sphero-conal system in the following manner.

Consider

$$r^2 = x^2 + y^2 + z^2 \tag{2.2a}$$

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2-b^2} - \frac{z^2}{c^2-\mu^2} = 0 \tag{2.2b}$$

$$\frac{x^2}{\nu^2} - \frac{y^2}{b^2-\nu^2} - \frac{z^2}{c^2-\nu^2} = 0 \tag{2.2c}$$

and the inverse transformation

$$x = \frac{r\mu\nu}{bc} \tag{2.3a}$$

$$y = \frac{r\sqrt{\mu^2-b^2}\ \sqrt{b^2-\nu^2}}{\sqrt{c^2-b^2}} \tag{2.3b}$$

$$z = \frac{r\sqrt{c^2-\mu^2}\ \sqrt{c^2-\nu^2}}{c\sqrt{c^2-b^2}} \tag{2.3c}$$

where

$$0 \leq \nu^2 \leq b^2$$

$$b^2 \leq \mu^2 \leq c^2$$

The surfaces r, μ, and ν are mentioned by Morse and Feshbach [7], Byerly [8], Moon and Spencer [9], and several other authors under the heading of conical coordinates.

In general, the surfaces r, μ, and ν intersect at eight points, which introduces an eight fold ambiguity in fixing a point in space. In order to achieve a one to one correspondence between (x, y, z) and (r, μ, ν), the variables θ and φ, and the parameter k are used. They are defined so that

$$\cos\theta = \nu/b \qquad 0 \leq \theta \leq \Pi \tag{2.4a}$$

$$\sin\varphi = \frac{\sqrt{\mu^2-b^2}}{\sqrt{c^2-b^2}} \qquad 0 \leq \varphi \leq 2\Pi \tag{2.4b}$$

$$k = b/c \tag{2.4c}$$

In terms of the new variables, equations (2.3) become equations (2.1). For a more detailed discussion of this "uniformization", see Kraus and Levine [1,6].

The surface $\theta = \theta_1$ is an elliptic cone with its axis the x axis and its tip at the origin. (See Figure 2). The angle between the x axis and the surface of the cone in the z = 0 plane is θ_1. The angle between the x axis and the line passing through the origin and one of the foci of the ellipse in the y = 0 plane at the plane x = constant is given by ϵ and is related to k by

$$k^2 = \frac{1}{1 + \tan^2 \epsilon \cos^2 \theta_1} \tag{2.5}$$

Note that for $\theta = 0$, the surface is a plane angular sector in the y = 0 plane, centered around the positive x axis. For $\theta = \Pi/2$, it is the entire x = 0 plane. For $\theta = \Pi$, it is again a plane angular sector in the y = 0 plane, but it is centered around the negative x axis.

When $\theta = \Pi$, ϵ is the semi-angle of the angular sector, and

$$k^2 = \frac{1}{1 + \tan^2 \epsilon} = \cos^2 \epsilon \quad . \tag{2.6}$$

Thus, $\theta = \Pi$, $k^2 = 1/2$ corresponds to the quarter plane shown in Figure 3. The quarter plane lies in the y = 0 plane and is symmetric around the negative x axis. The corner angle is $\Pi/2$. When $k^2 = 0$, $\epsilon = \Pi/2$, the corner angle becomes Π, and the plane angular sector is a half plane. When $k^2 = 1$, $\epsilon = 0$, and the plane angular sector reduces to the negative x axis.

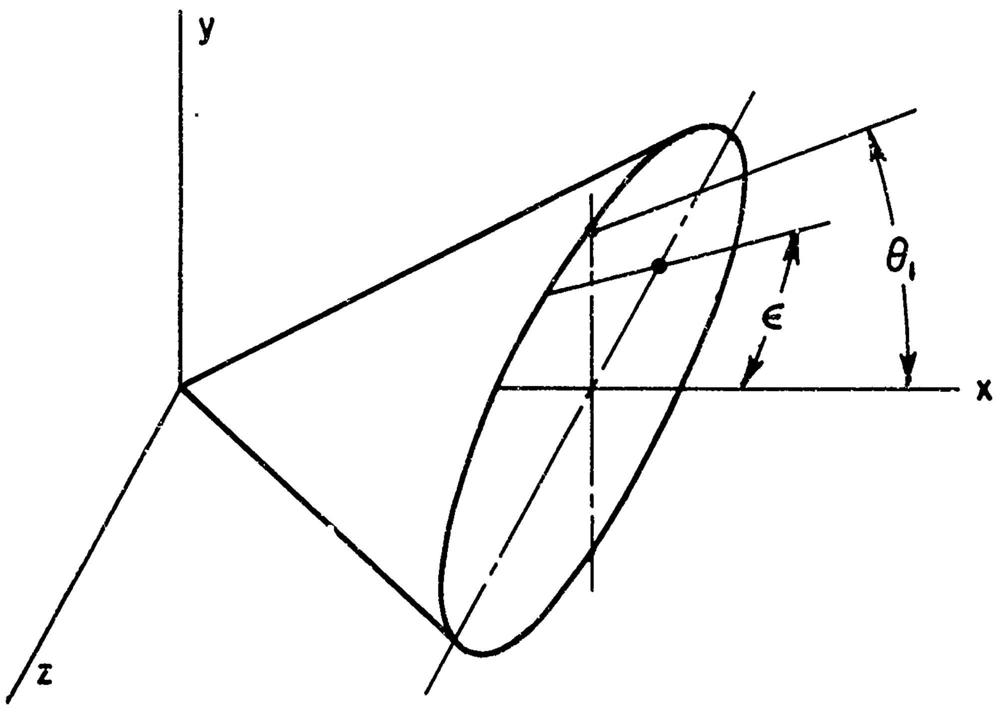

Fig. 2. The surface $\theta = \theta_1$

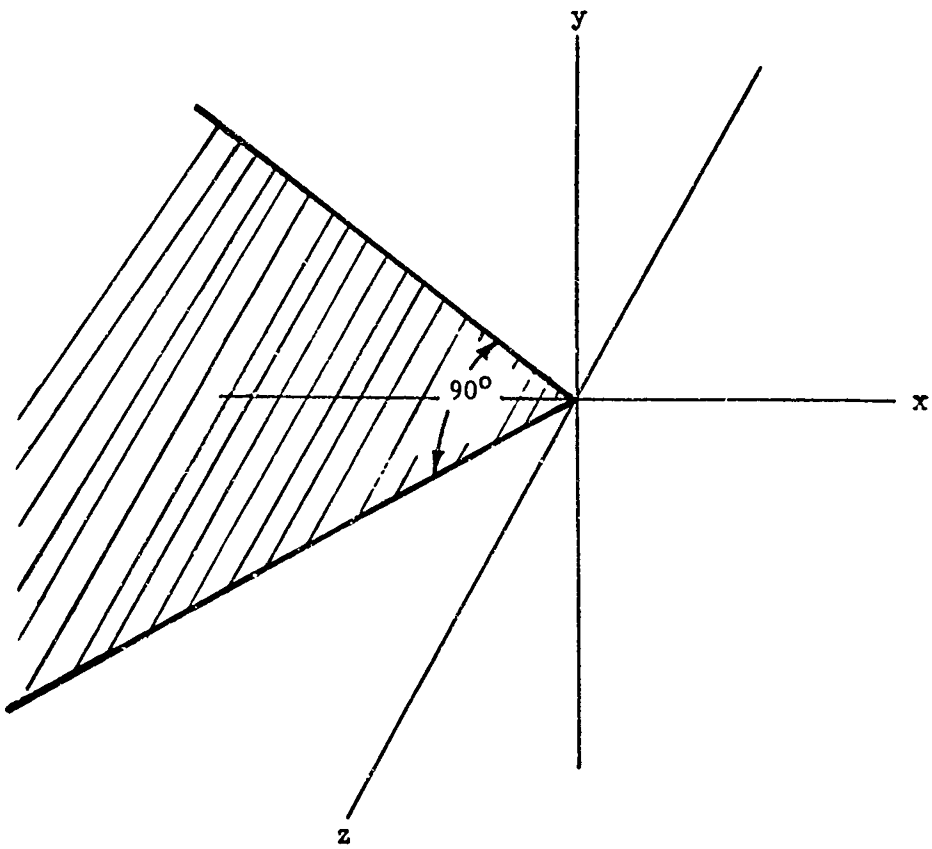

Fig. 3. The surface $\theta=\pi$, $k^2=1/2$

Note that for $k^2 = 1$, the surface $\theta = \theta_1$ is a right circular cone. Actually for this value of k^2 the coordinate system becomes the familiar spherical coordinate system.

Now consider the elliptic half cone sketched in solid lines in Figure 4; it is the surface $\varphi = \varphi_1$. The surfaces φ_1 and $2\Pi - \varphi_1$ compose a complete elliptic cone with its axis the z axis. Its characteristics are described in the same manner as the $\theta = \theta_1$ surface using the parameter k'^2. For $\varphi = 0$, the surface is a plane angular sector in the $y = 0$ plane, centered around the z axis. For $\varphi = \Pi/2$, it is the half plane $z = 0$, $y \geq 0$. For $\varphi = \Pi$, it is again a plane angular sector in the $y = 0$ plane, centered around the negative z axis. For $\varphi = 3\Pi/2$, it is the half plane $z = 0$, $y \leq 0$, and for $\varphi = 2\Pi$, it coincides with the $\varphi = 0$ surface.

The intersection of $\theta = \theta_1$ and $\varphi = \varphi_1$ is a line in the r direction. The intersection of this line with the sphere $r = r_1$ yields the point $(r_1, \theta_1, \varphi_1)$ as shown in Figure 5.

From equations (2.1) the unit vectors and metric coefficients are determined using standard techniques.

$$\hat{r} = \frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{z}{r}\hat{z} \tag{2.7a}$$

$$\hat{\theta} = \frac{-\sin\theta\sqrt{1-k^2\cos^2\theta}\ \sqrt{1-k'^2\cos^2\varphi}}{\sqrt{k^2\sin^2\theta + k'^2\sin^2\varphi}}\,\hat{x}$$

$$+ \frac{\cos\theta\sin\varphi\ \sqrt{1-k^2\cos^2\theta}}{\sqrt{k^2\sin^2\theta + k'^2\sin^2\varphi}}\,\hat{y} \tag{2.7b}$$

$$+ \frac{k^2\cos\theta\sin\theta\cos\varphi}{\sqrt{k^2\sin^2\theta + k'^2\sin^2\varphi}}\,\hat{z}$$

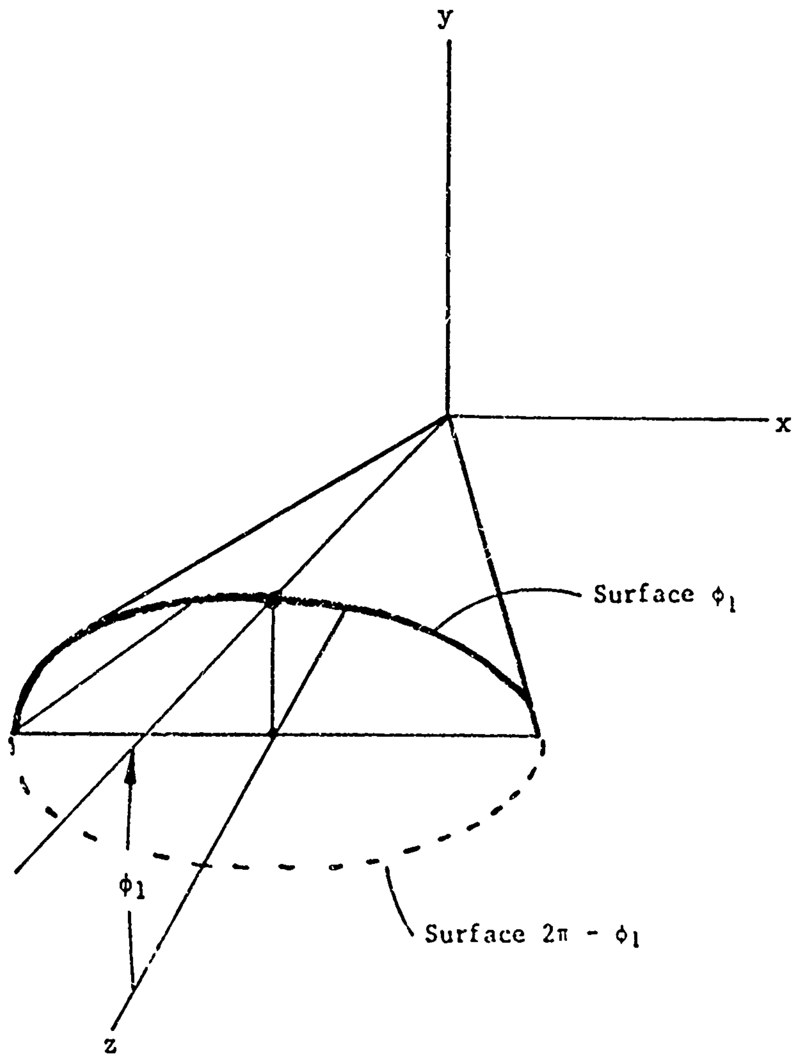

Fig. 4. The surface ϕ_1 and 2π-ϕ_1

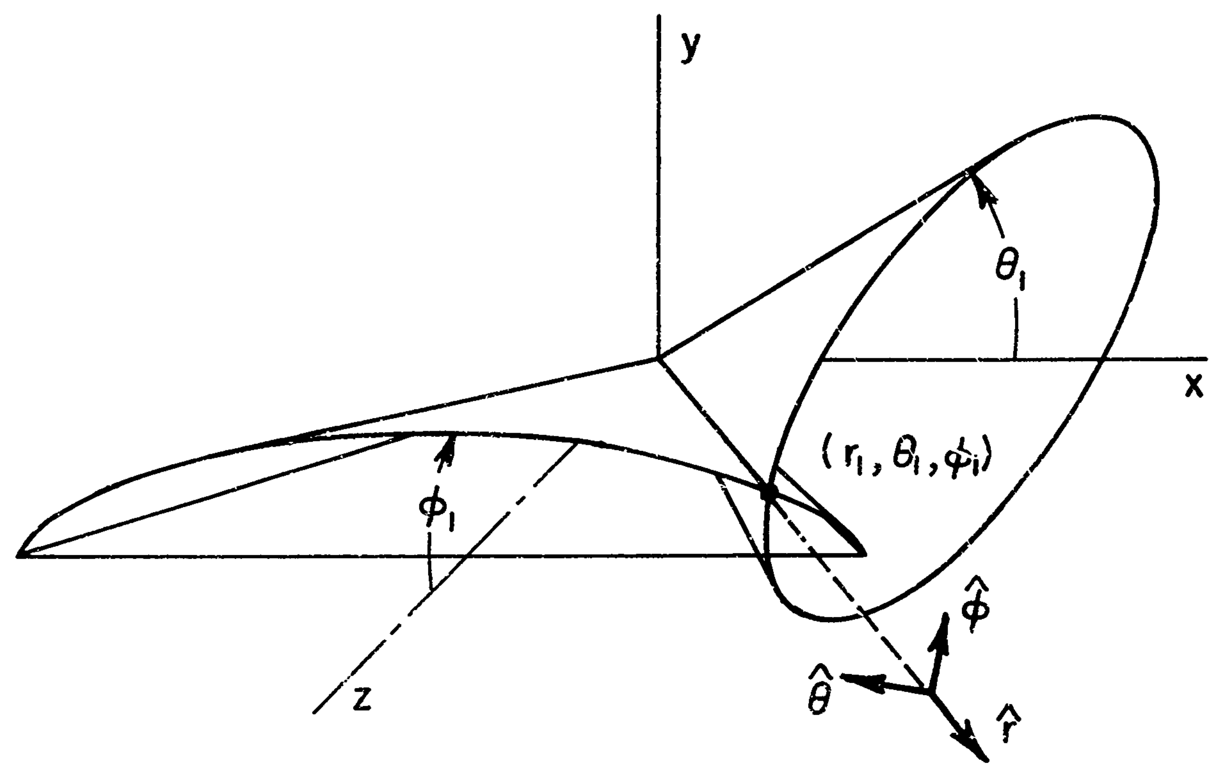

Fig. 5. Unit Vectors at (r_1, θ_1, ϕ_1)

$$\hat{\varphi} = k'^2 \cos\theta \cos\varphi \sin\varphi \quad \hat{x}$$

$$+ \frac{\sin\theta \cos\varphi \sqrt{1-k'^2 \cos^2\varphi}}{\sqrt{k^2 \sin^2\theta + k'^2 \sin^2\varphi}} \quad \hat{y} \tag{2.7c}$$

$$- \frac{\sin\varphi \sqrt{1-k^2 \cos\theta} \ \sqrt{1-k'^2 \cos^2\varphi}}{\sqrt{k^2 \sin^2\theta + k'^2 \sin^2\varphi}} \quad \hat{z}$$

The vectors $\hat{r}$, $\hat{\varphi}$, $\hat{\theta}$ form a right-handed orthogonal system with each vector pointing in the direction of the increasing coordinate. (See Figure 5.)

The metric coefficients are

$$h_r = 1 \tag{2.8a}$$

$$h_\theta = \frac{r\sqrt{k^2 \sin^2\theta + k'^2 \sin^2\varphi}}{\sqrt{1-k^2 \cos^2\theta}} \tag{2.8b}$$

$$h_\varphi = \frac{r\sqrt{k^2 \sin^2\theta + k'^2 \sin^2\varphi}}{\sqrt{1-k'^2 \cos^2\varphi}} \tag{2.8c}$$

Using equations (2.8), the gradient operator is

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\sqrt{1-k^2\cos^2\theta}}{r\sqrt{k^2\sin^2\theta + k'^2\sin^2\varphi}}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{\sqrt{1-k'^2\cos^2\varphi}}{r\sqrt{k^2\sin^2\theta + k'^2\sin^2\varphi}}\frac{\partial}{\partial\varphi} \tag{2.9}$$

and the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2(k^2\sin^2\theta + k'^2\sin^2\varphi)}\left[\sqrt{1-k^2\cos^2\theta}\,\frac{\partial}{\partial\theta}\left(\sqrt{1-k^2\cos^2\theta}\,\frac{\partial}{\partial\theta}\right) + \sqrt{1-k'^2\cos^2\varphi}\,\frac{\partial}{\partial\varphi}\left(\sqrt{1-k'^2\cos^2\varphi}\,\frac{\partial}{\partial\varphi}\right)\right] \tag{2.10}$$

The Effect of Peculiarities of the Coordinate System on Solutions of the Wave Equation

From the description of the φ coordinate surface in the previous section, it is seen that the variable φ is periodic with period 2Π. Thus in the absence of any physical boundaries in the φ direction it is necessary that solutions, $\psi(r, \theta, \varphi)$, of the wave equation satisfy the following periodicity condition.

$$\psi(r,\theta,\varphi) = \psi(r,\theta,\varphi + 2\Pi) \tag{2.11}$$

This is the same condition that occurs in cylindrical and spherical coordinate systems; it is a necessary condition if the solution is to be a single-valued function of φ.

Now consider the $y = 0$ plane. (See Figure 6). It is divided into four sectors which are the surfaces $\theta = 0$, $\theta = \Pi$, $\varphi = 0,2\Pi$, and $\varphi = \Pi$. The $\varphi = \Pi$ surface is regular and presents no difficulties. The $\varphi = 0,2\Pi$ surface is described in two different ways, but this is taken care of by the periodicity requirement just discussed. The $\theta = \Pi$ surface is the scattering body, and will have boundary conditions prescribed by the nature of the physical problem. The $\theta = 0$ surface is a singular coordinate surface. A point on it is described in two different ways depending on whether the surface is approached from above or below. For $y = 0^+$, a point is described by $(r,0,\varphi)$. For $y = 0^-$, the same point is given by $(r,0,2\Pi-\varphi)$. At $(r,0,\varphi)$, $\hat{\theta} = \hat{y}$, and $\hat{\varphi}$ has a negative z component. At $(r,0,2\Pi-\varphi)$, $\hat{\theta} = -\hat{y}$, and $\hat{\varphi}$ has a component in the positive z direction. This is easily seen by observing that the unit vectors

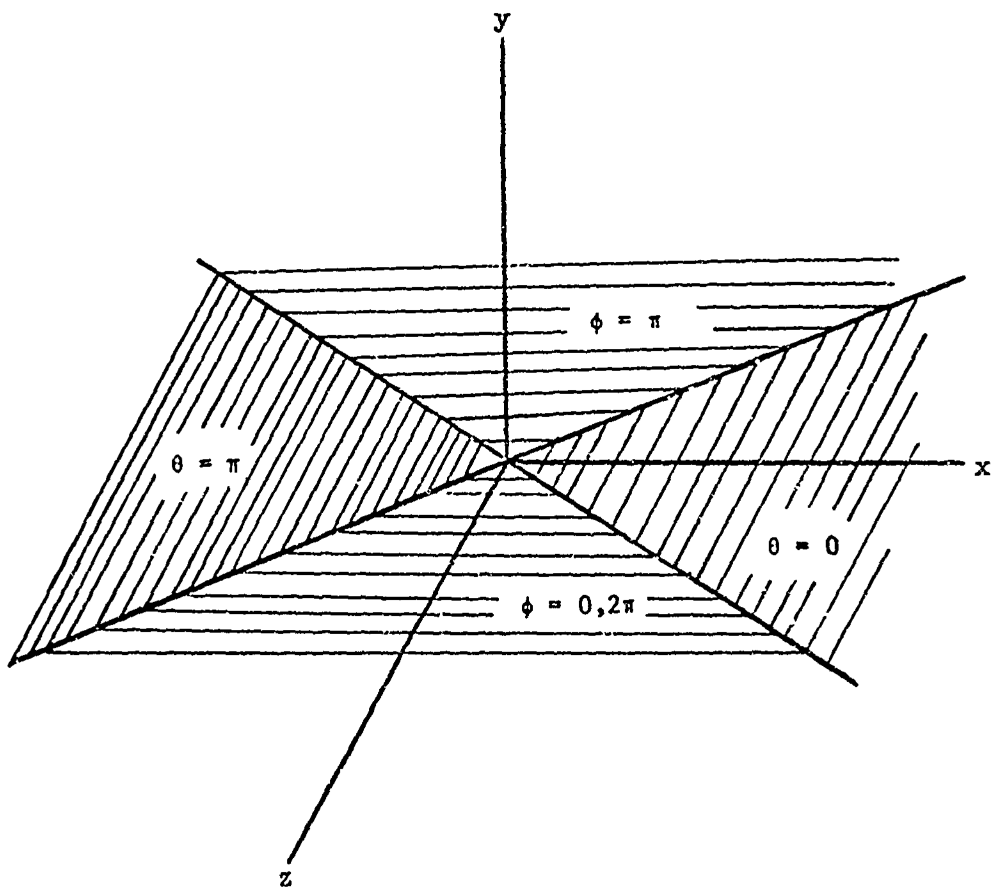

Fig. 6. Surfaces in the y=0 plane

point in the direction of the increasing coordinate. It is also evident from equations (2.7).

In order to have continuous scalar fields, it is necessary that $\psi(r,\theta,\varphi)$ and $\nabla\psi(r,\theta,\varphi)$ be continuous. For continuity of $\psi(r,\theta,\varphi)$ on the $\theta = 0$ surface.

$$\psi(r,0,\varphi) = \psi(r,0,2\Pi - \varphi) \tag{2.12}$$

Using the periodicity in φ, this can be written

$$\psi(r,0,\varphi) = \psi(r,0,-\varphi) \tag{2.13}$$

Inspection of equation (2.9) indicates that the following two equations must be satisfied for the gradient to be continuous at $\theta = 0$.

$$\frac{\partial\psi}{\partial\theta}(r,0,\varphi) = -\frac{\partial\psi}{\partial\theta}(r,0,-\varphi) \tag{2.14}$$

$$\frac{\partial\psi}{\partial\varphi}(r,0,\varphi) = -\frac{\partial\psi}{\partial\varphi}(r,0,-\varphi) \tag{2.15}$$

Equation (2.15) is a necessary consequence of equation (2.13), so it is not an independent equation.

Next consider the four lines described by ($\theta = \Pi$, $\varphi = 0, 2\Pi$), ($\theta = \Pi$, $\varphi = \Pi$), ($\theta = 0$, $\varphi = 0, 2\Pi$), and ($\theta = 0$, $\varphi = \Pi$). The two lines bordering $\theta = \Pi$ are part of the scattering surface and will have boundary conditions prescribed by the physical problem. Consider the line ($\theta = 0$, $\varphi = 0$). On the $\varphi = 0$ surface, $\hat{\varphi} = \hat{y}$ and $\hat{\theta}$ is in the $y = 0$ plane. On the $\theta = 0$ surface $\hat{\theta} = \hat{y}$ and $\hat{\varphi}$ is in the $y = 0$ plane. In order to have a continuous gradient at the line ($\theta = 0$, $\varphi = 0$), it is necessary that various components of the gradient behave properly as the line is approached from different directions. For example, consider the component of the gradient which is perpendicular to the $y = 0$ plane. On the $\theta = 0$ surface, it is given by $\nabla\psi \cdot \hat{\theta}$. On the $\varphi = 0$ surface it is given by $\nabla\psi \cdot \hat{\varphi}$. As the line $(\theta = 0, \varphi = 0)$ is approached, it is necessary that

$$\lim_{\substack{\theta \to 0^+ \\ \varphi = 0}} \nabla\psi \cdot \hat{\varphi} = \lim_{\substack{\varphi \to 0^+ \\ \theta = 0}} \nabla\psi \cdot \hat{\theta} \tag{2.16}$$

Kraus and Levine have investigated all of the necessary conditions on the components of the gradient and have determined that they are automatically satisfied for solutions of the wave equation. For details on this rather ingenious derivation, see their paper [6].

To summarize, the "boundary conditions" imposed by the coordinate system are given by equations (2.11), (2.13), and (2.14).

$$\psi(r,\theta,\varphi) = \psi(r,\theta,\varphi + 2\Pi) \tag{2.11}$$

$$\psi(r,0,\varphi) = \psi(r,0, -\varphi) \tag{2.13}$$

$$\frac{\partial\psi}{\partial\theta}(r,0,\varphi) = -\frac{\partial\psi}{\partial\theta}(r,0, -\varphi) \tag{2.14}$$

CHAPTER III

SOLUTION OF THE SCALAR WAVE EQUATION

This chapter contains a description of the scalar problem and the separation of the scalar wave equation into a spherical Bessel's equation and two coupled Lamé equations. The angular boundary values are discussed and it is seen that the Lamé problem can be decomposed into four boundary value problems, the solutions of which comprise two complete sets of functions. Each of these four problems is discussed in detail and then the solutions are tabulated.

Description of the Problem

As mentioned earlier, the primary purpose of this report is to determine a dyadic Green's function for the plane angular sector. In order to do this, solutions of the vector wave equation are needed, and in order to solve the vector wave equation, solutions of the scalar wave equation are needed. The solution of the scalar wave equation is described in this chapter.

The scalar wave equation is

$$(\nabla^2 + \kappa^2)\, \psi(r,\theta,\phi) = 0 \qquad (3.1)$$

where κ is the usual wave number. Using the expression for ∇^2 (equation 2.10) in the sphero-conal coordinate system, this becomes

$$\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right)\right) + \frac{1}{r^2\,(k^2 \sin^2\theta + k'^2 \sin^2\phi)} \tag{3.2}$$

$$\left[\sqrt{1 - k^2 \cos^2\theta}\,\frac{\partial}{\partial\theta}\left(\sqrt{1 - k^2\cos^2\theta}\,\frac{\partial\psi}{\partial\theta}\right)\right.$$

$$\left. + \sqrt{1 - k'^2 \cos^2\phi}\,\frac{\partial}{\partial\phi}\left(\sqrt{1 - k'^2\cos^2\phi}\,\frac{\partial\psi}{\partial\phi}\right)\right] + \kappa^2\,\psi = 0\ .$$

Using standard separation of variables techniques, the solution is written as

$$\psi(r,\theta,\phi) = R(r)\ \Theta(\theta)\ \Phi(\phi) \tag{3.3}$$

and substituted into equation (3.2) which separates into

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + (\kappa^2\, r^2 - \nu\,(\nu+1))\ R = 0 \tag{3.4}$$

and

$$\sqrt{1 - k^2 \cos^2\theta}\,\frac{\partial}{\partial\theta}\left(\sqrt{1 - k^2\cos^2\theta}\,\frac{\partial}{\partial\theta}\,(\Theta\Phi)\right) \tag{3.5}$$

$$+ \sqrt{1 - k'^2 \cos^2\phi}\,\frac{\partial}{\partial\phi}\left(\sqrt{1 - k'^2\cos^2\phi}\,\frac{\partial}{\partial\phi}\,(\Theta\Phi)\right)$$

$$+ \nu\,(\nu+1)\,(k^2 \sin^2\theta + k'^2 \sin^2\phi)\ \Theta\Phi = 0\ .$$

Equation (3.4) is the spherical Bessel equation with solutions

$$R(r) = \begin{cases} j_\nu\,(\kappa r) = \sqrt{\dfrac{\pi}{2\kappa r}}\ \ J_{\nu+1/2}\,(\kappa r) & \text{(3.6a)} \\[2ex] h_\nu^{(2)}\,(\kappa r) = \sqrt{\dfrac{\pi}{2\kappa r}}\ \ H^{(2)}_{\nu+1/2}\,(\kappa r) & \text{(3.6b)} \end{cases}$$

The $e^{+j\omega t}$ time convention is used here and thus $h_\nu^{(2)}(\kappa r)$ represents a wave that is propagating away from the coordinate source. The order of the Bessel function, ν, is determined from equation (3.5) and the boundary conditions on θ and ϕ.

Before taking a closer look at equation (3.5), the restrictions imposed by the coordinate system "boundary conditions" will be investigated. From equation (2.11)

$$\Phi(\phi) = \Phi(\phi + 2\pi) \tag{3.7}$$

From equation (2.13)

$$\Theta(0)\ \Phi(\phi) = \Theta(0)\ \Phi(-\phi) \tag{3.8}$$

From equation (2.14)

$$\Theta'(0)\ \Phi(\phi) = -\Theta'(0)\ \Phi(-\phi) \tag{3.9}$$

To see what the last two conditions mean, separate the functions $\Theta(\theta)$ and $\Phi(\phi)$ into even and odd parts. Since θ is restricted in the range $0 \leq \theta \leq \pi$, the terms even and odd here simply mean that $\Theta_e'(0) = 0$ and $\Theta_o(0) = 0$.

$$\Theta(\theta) = \Theta_e(\theta) + \Theta_o(\theta) \tag{3.10a}$$

$$\Phi(\phi) = \Phi_e(\phi) + \Phi_o(\phi) \tag{3.10b}$$

Using equations (3.10) and the properties of even and odd functions, equation (3.8) can be written

$$\Theta_e(0)\ \Phi_e(\phi) + \Theta_e(0)\ \Phi_o(\phi) = \Theta_e(0)\ \Phi_e(\phi) - \Theta_e(0)\ \Phi_o(\phi)\ . \tag{3.11}$$

This can be satisfied if

$$\Phi_o(\phi) = 0,\ \text{i.e.}\ \ \Phi(\phi)\ \text{is even} \tag{3.12}$$

or if

$$\Theta_e(0) = 0 \tag{3.13}$$

This last condition, along with the fact that $\Theta_\epsilon'(0) = 0$, implies that $\Theta_e(\theta) \equiv 0$. Thus equation (3.13) implies that

$$\Theta(\theta) \text{ is odd .} \tag{3.14}$$

Using equations (3.10) and the properties of even and odd functions, equation (3.9) can be written

$$\Theta_o'(0)\,\Phi_e(\phi) + \Theta_o'(0)\,\Phi_o(\phi) = -\,\Theta_o'(0)\,\Phi_e(\phi) + \Theta_o'(0)\,\Phi_o(\phi)\ . \tag{3.15}$$

This can be satisfied if

$$\Phi_e(\phi) = 0, \text{ i.e. } \Phi(\phi) \text{ is odd} \tag{3.16}$$

or if

$$\Theta_o'(0) = 0\ . \tag{3.17}$$

This last condition, along with the fact that $\Theta_o(0) = 0$, implies that $\Theta_o(\theta) \equiv 0$. Thus equation (3.17) implies that

$$\Theta(\theta) \text{ is even .} \tag{3.18}$$

In summary, equation (3.8) can be satisfied if $\Phi(\phi)$ is even or if $\Theta(\theta)$ is odd. Equation (3.9) can be satisfied if $\Phi(\phi)$ is odd or if $\Theta(\theta)$ is even. The only non-trivial combination of these conditions is both $\Theta(\theta)$ and $\Phi(\varphi)$ even or both $\Theta(\theta)$ and $\Phi(\phi)$ odd. Thus the coordinate imposed "boundary conditions" require the solution of equation (3.5) to be written in the form

$$\Theta(\theta)\,\Phi(\phi) = \begin{cases} \Theta_e(\theta)\,\Phi_e(\phi) \\ \\ \Theta_o(\theta)\,\Phi_o(\phi) \end{cases} \tag{3.19}$$

with $\Phi_e(\phi)$ and $\Phi_o(\phi)$ periodic with period 2π.

The physical boundary conditions are imposed at the plane angular sector and will be conditions on $\Theta(\theta)$ since the plane angular sector is described by $\theta = \pi$. It will be seen in the next chapter that the solution to the vector problem requires both Dirichlet and Neumann scalar solutions; that is, two types of $\Theta(\theta)$ functions are needed, those that satisfy $\Theta(\pi) = 0$, and those that satisfy $\Theta'(\pi) = 0$.

To completely specify the problem, it is necessary to normalize the functions. A convenient normalization is accomplished by setting

$$\Theta_e(0) = \Phi_e(0) = 1 \tag{3.20a}$$

and

$$\Theta_o'(0) = \Phi_o'(0) = 1 \, . \tag{3.20b}$$

Equation (3.5) can be separated into the following two equations

$$\sqrt{1 - k^2 \cos^2 \theta}\, \frac{d}{d\theta} \left(\sqrt{1 - k^2 \cos^2 \theta}\, \frac{d\Theta}{d\theta} \right)$$

$$+ \left(\nu(\nu+1)\, k^2 \sin^2 \theta + \mu\right) \Theta = 0 \tag{3.21}$$

$$\sqrt{1 - k'^2 \cos^2 \phi}\, \frac{d}{d\phi} \left(\sqrt{1 - k^2 \cos^2 \phi}\, \frac{d\Phi}{d\phi} \right)$$

$$+ \left(\nu(\nu+1)\, k'^2 \sin^2 \phi - \mu\right) \Phi = 0 \, . \tag{3.22}$$

Considering the boundary conditions, the solution of these equations is actually four separate problems.

I. Dirichlet

A. Even

Solve equations (3.21) and (3.22) subject to the boundary conditions

(1) $\Theta_{e1}(0) = 1$ $\qquad\qquad$ $\Theta_{e1}(\pi) = 0$

(2) $\Phi_{e1}(0) = 1$ $\qquad\qquad$ $\Phi_{e1}(\phi) = \Phi_{e1}(\phi + 2\pi)$

B. Odd

Solve equations (3.21) and (3.22) subject to the boundary conditions

(1) $\Theta_{o1}'(0) = 1$ $\Theta_{o1}(\pi) = 0$

(2) $\Phi_{o1}'(0) = 1$ $\Phi_{o1}(\phi) = \Phi_o(\phi + 2\pi)$

II. Neumann

A. Even

Solve equations (3.21) and (3.22) subject to the boundary conditions

(1) $\Theta_{e2}(0) = 1$ $\Theta_{e2}'(\pi) = 0$

(2) $\Phi_{e2}(0) = 1$ $\Phi_{e2}(\phi) = \Phi_{e2}(\phi + 2\pi)$

B. Odd

Solve equations (3.21) and (3.22) subject to the boundary conditions

(1) $\Theta_{o2}'(0) = 1$ $\Theta_{o2}'(\pi) = 0$

(2) $\Phi_{o2}'(0) = 1$ $\Phi_{o2}(\phi) = \Phi_{o2}(\phi + 2\pi)$

Before investigating each of these problems separately, some general results are discussed. First, note that each problem is two two-parameter Sturm-Liouville problems. Thus, for each value of ν in equations (3.21) and (3.22) there are an infinite number of μ's which can satisfy each equation. Kraus and Levine [6] have shown that only a finite number of μ's are needed for each ν in order to have a complete set of solutions. Also note that equation (3.5) is a two-dimensional Sturm-Liouville type equation. It can be shown that the two-dimensional Sturm-Liouville type operator is self adjoint and positive definite and, therefore, that the eigenvalues ν are all positive. These results will be used to find the eigenvalues ν and μ.

It follows from the self adjoint properties of the differential equations that the solutions to the Dirichlet and Neumann problems are orthogonal. The orthogonality relationship takes the form

$$\int_S \Theta_n(\theta)\ \Phi_n(\phi)\ \Theta_p(\theta)\ \Phi_p(\phi)\ dS = 0 \quad n \neq p \tag{3.23}$$

where the subscripts n and p indicate that the eigenfunctions correspond to the eigenvalue pairs (ν_n, μ_n) and (ν_p, μ_p). The surface S is spherical. It is understood that all of the eigenfunctions in equation (3.23) belong to either the Dirichlet set or the Neumann set. Dirichlet eigenfunctions are not necessarily orthogonal to Neumann eigenfunctions.

The self-adjoint property and the positive-definite property of the two-dimensional Sturm-Liouville type operator are proved in Appendix A. For proof of completeness and orthogonality, see Kraus and Levine[1,6].

Method of Solution

IA. Even Dirichlet Problem

The usual approach to an unfamiliar differential equation is to try a power series solution. By assuming a solution to equation (3.21) of the form

$$\Theta_{e1}(\theta) = \sum_{m=0}^{\infty} A_m \cos^m\theta/2$$

it is seen that there are two independent solutions, one with m even and the other with m odd. By imposing the Dirichlet boundary condition, the solution with m even can be eliminated. Instead of writing the solution in the form

$$\Theta_{e1}(\theta) = \sum_{m=1,3,\ldots} A_m \cos^m \theta/2$$

it is found to be more convenient to use

$$\Theta_{e1}(\theta) = \sum_m A_m \cos\,(2m - 1/2)\,\theta \tag{3.24}$$

where the summation is over all m. The recurrence relation for equation (3.24) is

$$A_{m-1} \quad \frac{k^2}{4} \left(\frac{(4m-3)(4m-5)}{4} - \nu(\nu+1) \right) \tag{3.25}$$

$$+ A_m \left(\frac{(4m-1)^2}{4} \left(\frac{k^2}{2} - 1\right) + \frac{\nu(\nu+1)k^2}{2} + \mu \right)$$

$$+ A_{m+1} \quad \frac{k^2}{4} \left(\frac{(4m+1)(4m+3)}{4} - \nu(\nu+1) \right) = 0$$

This set of equations can be written in matrix form, and a determinant can be identified which must be zero in order to have a non-trivial solution. Ince[10] has considered a similar problem which must have the same solution as this problem. He sets up the problem in the same way and identifies an infinite determinant which must be zero. With some rather straightforward manipulations. an infinite determinant of the type considered here can be written in the form of an infinite continued fraction. The fraction for this problem is

$$h = \frac{1+k^2}{4} + \frac{(2\nu-1)(2\nu+3)\,k^2/9}{1+k^2-4h/9} + \frac{16\,(2\nu-3)(2\nu+5)\,k^2/225}{1+k^2-4h/25}$$

$$+ \frac{36\,(2\nu-5)(2\nu+7)\,k^2/1225}{1+k^2-4h/49} + \ldots\ldots \tag{3.26}$$

where

$$\mu = h - \nu(\nu+1)\,k^2 \,. \tag{3.27}$$

Next, a solution to equation (3.22) is assumed. It is found that there are two independent solutions that can be written in the form

$$\Phi_{e1}(\phi) = \sum_{m=0}^{\infty} B_{2m} \cos 2m\,\phi \tag{3.28}$$

$$\Phi_{e1}(\phi) = \sum_{m=0}^{\infty} B_{2m+1} \cos (2m+1)\,\phi \tag{3.29}$$

The recurrence relations for equation (3.28) are

$$B_0 \left(\frac{\nu(\nu+1)\, k'^2}{2} - \mu\right) + B_2 \frac{k'^2}{4} (2 - \nu(\nu+1)) = 0 \tag{3.30a}$$

$$B_0 \left(\frac{-\nu(\nu+1)\, k'^2}{2}\right) + B_2 \left(4 \left(\frac{k'^2}{2} - 1\right) + \frac{\nu(\nu+1)\, k'^2}{2} - \mu\right)$$

$$+ B_4 \frac{k'^2}{4} \; (12 - \nu(\nu+1)) = 0 \tag{3.30b}$$

$$B_{2m-2} \; \frac{k'^2}{4} ((2m-2)\;(2m-1) - \nu(\nu+1)) \tag{3.30c}$$

$$+ B_{2m} \; ((2m)^2 \left(\frac{k'^2}{2} - 1\right) + \frac{\nu(\nu+1)\, k'^2}{2} - \mu)$$

$$+ B_{2m+2} \; \frac{k'^2}{4} \; ((2m+2)(2m+1) - \nu(\nu+1)) = 0 \qquad m \geq 2$$

and for equation (3.29)

$$B_{2m-1} \frac{k'^2}{4} ((2m-1)\;(2m) - \nu(\nu+1)) \tag{3.31}$$

$$+ B_{2m+1} \; ((2m+1)^2 \left(\frac{k'^2}{2} - 1\right) + \frac{\nu(\nu+1)\, k'^2}{2} - \mu)$$

$$+ B_{2m+3} \frac{k'^2}{4} \; ((2m+3)\;(2m+2) - \nu(\nu+1)) = 0$$

As before, these equations can be written in matrix form, and the determinant which must be zero can be identified. The determinantal equations can be written

$$\eta = \cfrac{-(\nu^2-1)(\nu)(\nu+2)\ k'^4/8}{2 - k'^2 - \eta/4} - \cfrac{(\nu^2-9)(\nu-2)(\nu+4)\ k'^4/(16)^2}{2 - k'^2 - \eta/16}$$

$$- \cfrac{(\nu^2-25)(\nu-4)(\nu+6)\ k'^4/(48)^2}{2 - k'^2 - \eta/36} - \cdots\cdots \tag{3.32}$$

for equations (3.30) and

$$\eta = \nu(\nu+1)\ k'^2/2 + 2 - k'^2 - \cfrac{(\nu^2-4)(\nu-1)(\nu+3)\ k'^4/(6)^2}{2 - k'^2 - \eta/9}$$

$$- \cfrac{(\nu^2-16)(\nu-3)(\nu+5)\ k'^4/(30)^2}{2 - k'^2 - \eta/25} - \cdots\cdots \tag{3.33}$$

for equation (3.31) where

$$\mu = 1/2\ (-\eta + \nu(\nu+1)\ k'^2)\ . \tag{3.34}$$

In order to find the eigenvalues for problem IA, it is necessary to solve equations (3.26) and (3.32), and equations (3.26) and (3.33) simultaneously. The solutions are the eigenvalues (ν,μ) of the even Dirichlet problem. The manner in which this is done is shown in Appendix B. The eigenvalues are tabulated for the quarter plane problem (k^2=1/2) in Table 1 (pages 41 to 44).

Once the eigenvalues are determined, the eigenfunctions can be found by solving the recurrence relations. This again involves the use of continued fractions and is explained in Appendix C. The form of the eigenfunctions is shown in Table 1, and the values of the coefficients are given in Table 2 (pages 45 to 56).

IB. Odd Dirichlet Problem

Solutions to this problem may be expressed in terms of the standard Lamé polynomials. The eigenvalues ν are integers, and there are exactly ν μ's for each value of ν. The solutions can be found in the same way as in problem IA. Series solutions are assumed, and it is found that the series are finite for integral values of ν if they are written in the following forms. For equation (3.21),

$$\Theta_{01}(\theta) = \sum_{m=0}^{N} A_{2m+1} \sin (2m+1)\, \theta \tag{3.35}$$

$$\Theta_{01}(\theta) = \sum_{m=1}^{N} A_{2m} \sin 2m\, \theta \tag{3.36}$$

$$\Theta_{01}(\theta) = \sqrt{1 - k^2 \cos^2 \theta} \sum_{m=0}^{N} A_{2m+1} \sin (2m+1)\, \theta \tag{3.37}$$

$$\Theta_{01}(\theta) = \sqrt{1 - k^2 \cos^2 \theta} \sum_{m=1}^{N} A_{2m} \sin 2m\, \theta \tag{3.38}$$

where N is an integer that depends on the eigenvalue ν. Equations (3.35) and (3.37) describe one independent solution; equations (3.36) and (3.38) describe the other independent solution.

Each of these solutions gives rise to a recurrence relation and a continued fraction equation for the eigenvalues. The recurrence relations for equation (3.35) are

$$A_1 \left(\left(\frac{k^2}{2} - 1\right) + \frac{3\nu(\nu+1)\, k^2}{4} + \mu \right) + A_3 \frac{k^2}{4} (6 - \nu(\nu+1)) = 0 \tag{3.39a}$$

$$\begin{aligned} &A_{2m-1} \frac{k^2}{4} ((2m-1)\,(2m) - \nu(\nu+1)) \\ &+ A_{2m+1} \left((2m+1)^2 \left(\frac{k^2}{2} - 1\right) + \frac{\nu(\nu+1)\, k^2}{2} + \mu \right) \\ &+ A_{2m+3} \frac{k^2}{4} ((2m+3)(2m+2) - \nu(\nu+1)) = 0 \qquad m \geq 1 \end{aligned} \tag{3.39b}$$

The recurrence relation for equation (3.36) is

$$A_{2m-2}\,\frac{k^2}{4}\,((2m-2)(2m-1)-\nu(\nu+1)) \tag{3.40}$$

$$+\,A_{2m}\,\left((2m)^2\left(\frac{k^2}{2}-1\right)+\frac{\nu(\nu+1)\,k^2}{2}+\mu\right)$$

$$+\,A_{2m+2}\,\frac{k^2}{4}\,((2m+2)(2m+1)-\nu(\nu+1))=0 \qquad m\geq 1,\quad A_0=0$$

The recurrence relations for equation (3.37) are

$$A_1\left(\left(\frac{k^2}{2}-1\right)+\frac{3\nu(\nu+1)\,k^2}{4}+\mu\right)+A_3\,\frac{k^2}{4}\,(2-\nu(\nu+1))=0 \tag{3.41a}$$

$$A_{2m-1}\,\frac{k^2}{4}\,((2m+1)(2m)-\nu(\nu+1)) \tag{3.41b}$$

$$+\,A_{2m+1}\,\left((2m+1)^2\left(\frac{k^2}{2}-1\right)+\frac{\nu(\nu+1)\,k^2}{2}+\mu\right)$$

$$+\,A_{2m+3}\,\frac{k^2}{4}\,((2m+1)(2m+2)-\nu(\nu+1))=0 \qquad m\geq 1$$

and the recurrence relation for equation (3.38) is

$$A_{2m-2}\,\frac{k^2}{4}\,((2m)(2m-1)-\nu(\nu+1)) \tag{3.42}$$

$$+\,A_{2m}\,\left((2m)^2\left(\frac{k^2}{2}-1\right)+\frac{\nu(\nu+1)k^2}{2}+\mu\right)$$

$$+\,A_{2m+2}\,\frac{k^2}{4}\,((2m)(2m+1)-\nu(\nu+1))=0 \qquad m\geq 1,\quad A_0=0\,.$$

Each of these recurrence relations can be written in matrix form and the corresponding determinantal equations found. These equations written in the form of continued fractions are, for equation (3.35) and equation (3.37),

$$\eta = -\nu(\nu+1)\, k^2/2 + 2 - k^2 - \cfrac{(\nu^2-4)(\nu-1)(\nu+3)\, k^4/(6)^2}{2 - k^2 - \eta/9 - \cfrac{(\nu^2-16)(\nu-3)(\nu+5)\, k^4/(30)^2}{2 - k^2 - \eta/25 - \cdots}} \tag{3.43}$$

where

$$\mu = 1/2\,(\eta - \nu(\nu+1)\, k^2) \tag{3.44}$$

and for equation (3.36) and (3.38),

$$\eta = 4(2-k^2) - \cfrac{(\nu^2-9)(\nu-2)(\nu+4)\, k^4/(8)^2}{2 - k^2 - \eta/16 - \cfrac{(\nu^2-25)(\nu-4)(\nu+6)\, k^4/(48)^2}{2 - k^2 - \eta/36 - \cdots}} \tag{3.45}$$

where μ is again given by equation (3.44).

The same forms of solution are employed in equation (3.22) with the coefficients B instead of A.

$$\Phi_{01}(\phi) = \sum_{m=0}^{N} B_{2m+1} \sin (2m+1)\, \phi \tag{3.46}$$

$$\Phi_{01}(\phi) = \sum_{m=1}^{N} B_{2m} \sin 2m\, \phi \tag{3.47}$$

$$\Phi_{01}(\phi) = \sqrt{1 - k'^2 \cos^2 \phi}\, \sum_{m=0}^{N} B_{2m+1} \sin (2m+1)\, \phi \tag{3.48}$$

$$\Phi_{01}(\phi) = \sqrt{1 - k'^2 \cos^2 \phi}\, \sum_{m=0}^{N} B_{2m} \sin 2m\, \phi\ . \tag{3.49}$$

The recurrence relations are the same with k^2 replaced by k'^2 and μ replaced by $-\mu$. The eigenvalue equations are the same except that k^2 is replaced by k'^2. The definition of μ is just the negative of equation (3.44).

$$\mu = 1/2\,(-\eta + \nu(\nu+1)\, k'^2) \tag{3.50}$$

The eigenvalues are found by simultaneously solving equations (3.44) and (3.50). As previously mentioned, the eigenvalues ν are integers, and there are ν μ's for each value of ν. For the actual method of computation, see Appendix B. The eigenvalues are given in Table 1.

With the eigenvalues determined, the eigenfunctions are found by using the recurrence relations. This is done in Appendix C. The form of the eigenfunctions is shown in Table 1, and the values of the coefficients are given in Table 2.

IIA. Even Neumann Problem

Solutions to this problem are also standard Lamé polynomials. The eigenvalues ν are integers, and there are $\nu+1$ μ's for each value of ν. The even Neumann solutions can be found in the same way as the previous solutions. The solutions are expressed as suitable series, and it is found that the series are finite for integral values of ν if they are written in the following forms

$$\Theta_{e2}(\theta) = \sum_{m=0}^{N} A_{2m+1} \cos (2m+1) \theta \qquad (3.51)$$

$$\Theta_{e2}(\theta) = \sum_{m=0}^{N} A_{2m} \cos 2m \theta \qquad (3.52)$$

$$\Theta_{e2}(\theta) = \sqrt{1 - k^2 \cos^2 \theta} \sum_{m=0}^{N} A_{2m+1} \cos (2m+1) \theta \qquad (3.53)$$

$$\Theta_{e2}(\theta) = \sqrt{1 - k^2 \cos^2 \theta} \sum_{m=0}^{N} A_{2m} \cos 2m \theta \qquad (3.54)$$

where again N is an integer that depends on the value of ν.

With a few exceptions, the recurrence relations are the same as for the odd Dirichlet problem. For equation (3.51), the recurrence relation is

$$A_{2m-1} \frac{k^2}{4} ((2m-1)(2m) - \nu(\nu+1)) \tag{3.55}$$

$$+ A_{2m+1} ((2m+1)^2 (\frac{k^2}{2} -1) + \frac{\nu(\nu+1)\ k^2}{2} + \mu)$$

$$+ A_{2m+3} \frac{k^2}{4} ((2m+3)(2m+2) - \nu(\nu+1)) = 0$$

$$A_{-1} = 0, \quad m \geq 0$$

For equation (3.52), the recurrence relations are

$$A_0 (\frac{\nu(\nu+1)\ k^2}{2} + \mu) + A_2 \frac{k^2}{4} (2 - \nu(\nu+1)) = 0 \tag{3.56a}$$

$$A_0 (\frac{-\nu(\nu+1)\ k^2}{2}) + A_2 (4(\frac{k^2}{2} -1) + \frac{\nu(\nu+1)\ k^2}{2} + \mu) \tag{3.56b}$$

$$+ A_4 \frac{k^2}{4} (12 - \nu(\nu+1)) = 0$$

$$A_{2m-2} \frac{k^2}{4} ((2m-2)(2m-1) - \nu(\nu+1)) \tag{3.56c}$$

$$+ A_{2m} ((2m)^2 (\frac{k^2}{2} -1) + \frac{\nu(\nu+1)\ k^2}{2} + \mu)$$

$$+ A_{2m+2} \frac{k^2}{4} ((2m+2)(2m+1) - \nu(\nu+1)) = 0$$

For equation (3.53), the recurrence relation is

$$A_{2m-1} \frac{k^2}{4} ((2m+1)(2m) - \nu(\nu+1)) \tag{3.57}$$

$$+ A_{2m+1} ((2m+1)^2 (\frac{k^2}{2} -1) + \frac{\nu(\nu+1)\ k^2}{2} + \mu)$$

$$+ A_{2m+3} \frac{k^2}{4} ((2m+1)(2m+2) - \nu(\nu+1)) = 0$$

$$A_{-1} = 0, \quad m \geq 0$$

For equation (3.54), the recurrence relations are

$$A_0 (\frac{\nu(\nu+1)\ k^2}{2} + \mu) + A_2 (\frac{-\nu(\nu+1)\ k^2}{4}) = 0 \tag{3.58a}$$

$$A_0 \frac{k^2}{2} (2 - \nu(\nu+1)) + A_2 (4(\frac{k^2}{2} -1) + \frac{\nu(\nu+1)\ k^2}{2} + \mu) \tag{3.58b}$$

$$+ A_4 \frac{k^2}{4} (6 - \nu(\nu+1)) = 0$$

$$A_{2m-2} \frac{k^2}{4} ((2m)(2m-1) - \nu(\nu+1)) \tag{3.58c}$$

$$+ A_{2m} ((2m)^2 (\frac{k^2}{2} -1) + \frac{\nu(\nu+1)\ k^2}{2} + \mu)$$

$$+ A_{2m+2} \frac{k^2}{4} ((2m)(2m+1) - \nu(\nu+1)) = 0 \qquad m \geq 2$$

As before, each of these recurrence relations can be written in matrix form and determinantal equations found. These equations, written in the form of continued fractions are, for equations (3.51) and (3.53),

$$\eta = \nu(\nu+1)\ k^2/2 + 2 - k^2 - \frac{(\nu^2 -4)(\nu-1)(\nu+3)\ k^4/(6)^2}{2 - k^2 - \eta/9}$$

$$- \frac{(\nu^2 -16)(\nu-3)(\nu+5)\ k^4/(30)^2}{2 - k^2 - \eta/25} - \ldots\ldots \tag{3.59}$$

and for equations (3.52) and (3.54),

$$\eta = \frac{-(\nu^2-1)(\nu)(\nu+2)\ k^4/8}{2 - k^2 - \eta/4 - \dfrac{(\nu^2-9)(\nu-2)(\nu+4)\ k^4/(16)^2}{2 - k^2 - \eta/16 - \dfrac{(\nu^2-25)(\nu-4)(\nu+6)\ k^4/(48)^2}{2 - k^2 - \eta/36 - \cdots}}} \tag{3.60}$$

where

$$\mu = 1/2\ (\eta - \nu(\nu+1)\ k^2)\ . \tag{3.61}$$

Note that these are the same as equations (3.33) and (3.32) except for k^2 and k'^2.

The same form of solutions is assumed for equation (3.22) with the coefficients B instead of A.

$$\Phi_{e2}(\phi) = \sum_{m=0}^{N} B_{2m+1} \cos\ (2m+1)\ \phi \tag{3.62}$$

$$\Phi_{e2}(\phi) = \sum_{m=0}^{N} B_{2m} \cos\ 2m\ \phi \tag{3.63}$$

$$\Phi_{e2}(\phi) = \sqrt{1 - k'^2 \cos^2 \phi}\ \sum_{m=0}^{N} B_{2m+1} \cos\ (2m+1)\ \phi \tag{3.64}$$

$$\Phi_{e2}(\phi) = \sqrt{1 - k'^2 \cos^2 \phi}\ \sum_{m=0}^{N} B_{2m} \cos\ 2m\ \phi \tag{3.65}$$

The recurrence relations are the same with k^2 replaced by k'^2 and μ replaced by $-\mu$. The eigenvalue equations are the same except that k^2 is replaced by k'^2. The definition of μ is

$$\mu = 1/2\ (-\eta + \nu(\nu+1)\ k'^2) \tag{3.66}$$

The eigenvalues ν and μ are found in the same way and are tabulated in Table 3 (pages 57 to 60). The form of the eigenfunctions is also shown in Table 3, and the values of the coefficients are given in Table 4 (pages 61 to 72).

Together with the solutions of the odd Dirichlet problem, the eigenfunctions compose a complete set of functions that correspond to the Legendre polynomials in the spherical coordinate system. In fact, as the parameter k^2 approaches 1, the sphero-conal coordinate system degenerates into the spherical coordinate system, and the Lamé polynomials degenerate into Legendre polynomials. For each value $\nu = n$ there are 2n+1 standard Lamé polynomials. In principle, the solution to this problem could be found entirely in terms of these polynomials, since they are complete and can be used to represent any piecewise continuous function. In actual practice, the difficulties encountered because of the boundary conditions would have made this approach intractable. The Lamé polynomials have been studied by several authors. They are discussed by Ince[10,11], Erdelyi[12], Prasad[13], and several others, and are tabulated as power series by Arscott[14].

IIB. Odd Neumann Problem

This problem is very similar to the even Dirichlet problem. A series solution to equation (3.21) is assumed, and it is found that the one independent solution can be written in the form

$$\Theta_{O2}(\theta) = \sum_m A_m \sin (2m - 1/2) \theta \tag{3.67}$$

The recurrence relation is given by equation (3.25) and the eigenvalue equation by equation (3.26). Two independent solutions of equation (3.22) are found in the form

$$\Phi_{02}(\phi) = \sum_{m=1}^{\infty} B_{2m} \sin 2m\, \phi \tag{3.68}$$

$$\Phi_{02}(\phi) = \sum_{m=0}^{\infty} B_{2m+1} \sin (2m+1)\, \phi \tag{3.69}$$

The recurrence relations and eigenvalue equations are the same as for equations (3.36) and (3.35) with k^2 replaced by k'^2 and μ replaced by $-\mu$. The simultaneous solution of the eigenvalue equations yields the eigenvalues. The computational method is the same and is shown in Appendix B. The eigenvalues are tabulated in Table 3. The eigenfunction coefficients are determined in Appendix C and are tabulated in Table 4. The odd Neumann problem was also solved using a variational method; this is described in Appendix D. The first few eigenvalues and eigenfunctions were determined and compared with the solutions obtained by the more exact method. The comparison is reasonably good and is shown in Appendix D.

Discussion of Tables

Tables 1 through 4 show all of the eigenvalues and eigenfunctions for ν less than 9, and $k^2 = 1/2$. In Table 1, both the even and odd Dirichlet eigenvalues are tabulated, as well as the form of the corresponding eigenfunctions. Note the grouping of the eigenvalues. If n is an integer defined such that $n - 1/2 \leq \nu \leq n + 1/2$, there are $2n + 1$ μ's for each ν. Also note the pattern that is forming for the values of ν. As ν increases, the value of ν corresponding to large positive values of μ is approaching $n + 1/2$. As μ decreases for any one group of eigenvalues, the value of ν decreases until for large negative μ, ν is approaching n. The same type of pattern is noticed in Table 3 where both the even and odd Neumann eigenvalues are tabulated. For the Neumann eigenvalues, there are again $2n + 1$ μ's for each ν; ν approaches $n - 1/2$ for large positive μ, and n for large negative μ. Also note the pattern forming for the eigenvalues of the odd Dirichlet and even Neumann problems (Lamé polynomials). As ν increases, the eigenvalues for these two problems are beginning to coincide.

In Table 4, one should be careful when interpreting the coefficients of $\sum A_m \sin (2m - 1/2)\,\theta$. Coefficients with zero and negative subscripts correspond to sine functions with negative angles. Thus if the eigenfunctions are expanded in series of sines with positive angles, it is necessary to reverse the signs of all coefficients with zero and negative subscripts. This problem does not arise in the $\sum A_m \cos (2m - 1/2)\,\theta$ eigenfunctions since the cosine function is even.

The eigenfunctions in Tables 2 and 4 are normalized such that $\Theta_e(0) = \Phi_e(0) = 1$ and $\Theta_o'(0) = \Phi_o'(0) = 1$. Coefficients with magnitudes less than 5×10^{-4} are not given. Eigenvalues should be accurate to within $\pm\, 5 \times 10^{-3}$ even in the cases of least accuracy. This accuracy is not limited by the method and can be improved if desired. Since some of the eigenfunction coefficients are very sensitive to inaccuracies in the eigenvalues, it is expected that their accuracy is not as good. The continued fraction method used to find the eigenfunction coefficients, explained in Appendix C, tends to minimize this problem. In the case of the Lamé polynomials, the continued fraction method is not used, and it is expected that some of the coefficients with large subscripts are not precise. This occurs because the computation of the large subscripted coefficients usually involves the subtraction of two large numbers in order to determine a small number. These and other computational problems are discussed in the appendices.

Two other eigenvalues were computed for different values of k^2. The lowest eigenvalues were computed for $k^2 = 0.1$, corresponding to a plane angular sector with a corner angle of 143.14°, and for $k^2 = 0.9$, corresponding to a plane angular sector with a corner angle of 36.86°. These eigenvalues are given in Table 5 (page 73). The dominant behavior of the vector fields near the tip of the plane angular sector is governed by these eigenvalues; this is discussed in Chapter VI.

All of the computations for the eigenvalues and eigenfunctions were done on the IBM 360/75 computer.

Table 1 - Eigenvalues of the Dirichlet Problem and the Form of the Angular Eigenfunctions

$F = \sum A_m \cos (2m - 1/2)\, \theta$

$I = \sum A_{2m} \sin 2m\, \theta$

$J = \sum A_{2m+1} \sin (2m + 1)\, \theta$

$L = \sqrt{1-k^2 \cos^2 \theta} \sum A_{2m} \sin 2m\, \theta$

$P = \sqrt{1-k^2 \cos^2 \theta} \sum A_{2m+1} \sin (2m + 1)\, \theta$

$g = \sum B_{2m} \cos 2m\, \varphi$

$h = \sum B_{2m+1} \cos (2m + 1)\, \varphi$

$i = \sum B_{2m} \sin 2m\, \varphi$

$j = \sum B_{2m+1} \sin (2m + 1)\, \varphi$

$l = \sqrt{1-k'^2 \cos^2 \varphi} \sum B_{2m} \sin 2m\, \varphi$

$p = \sqrt{1-k'^2 \cos^2 \varphi} \sum B_{2m+1} \sin (2m + 1)\, \varphi$

ν	μ	Parity	$\Theta(\theta)$	$\Phi(\varphi)$
0.296	0.090	E	F	g
1.425	0.915	E	F	g
1.000	0.000	O	J	j
1.130	-0.455	E	F	h
2.480	2.670	E	F	g
2.000	1.500	O	I	p
2.290	0.215	E	F	h
2.000	-1.500	O	P	i
2.040	-1.705	E	F	g
3.495	5.440	E	F	g
3.000	3.873	O	J	j
3.410	1.535	E	F	h
3.000	0.000	O	L	l
3.145	-0.825	E	F	g
3.000	3.873	O	J	j
3.010	-3.940	E	F	h
4.499	9.225	E	F	g
4.000	7.190	O	I	p

Table 1 - (continued)

ν	μ	Parity	$\Theta(\theta)$	$\Phi(\varphi)$
4.470	3.790	E	F	h
4.000	2.190	O	P	i
4.280	0.335	E	F	g
4.000	- 2.190	O	I	p
4.050	- 2.575	E	F	h
4.000	- 7.190	O	P	i
4.005	- 7.205	E	F	g
5.500	14.013	E	F	g
5.000	11.489	O	J	j
5.490	7.110	E	F	h
5.000	5.196	O	L	l
5.499	2.110	E	F	g
5.000	0.000	O	J	j
5.150	- 1.170	E	F	h
5.000	- 5.196	O	L	l
5.015	- 5.335	E	F	g
5.000	-11.489	O	J	j
5.000	-11.493	E	F	h
6.500	19.805	E	F	g
6.000	16.783	O	I	p
6.499	11.455	E	F	h
6.000	9.114	O	P	i
6.465	4.850	E	F	g
6.000	2.832	O	I	p
6.282	0.445	E	F	h
6.000	- 2.832	O	P	i
6.055	- 3.380	E	F	g
6.000	- 9.114	O	I	p
6.005	- 9.160	E	F	h
6.000	-16.783	O	P	i
6.000	-16.873	E	F	g

Table 1 - (continued)

ν	μ	Parity	$\Theta(\theta)$	$\Phi(\varphi)$
7.500	26.597	E	F	g
7.000	23.075	O	J	j
7.500	16.820	E	F	h
7.000	14.000	O	L	l
7.492	8.695	E	F	g
7.000	6.444	O	J	j
7.393	2.660	E	F	h
7.000	0.000	O	L	l
7.157	- 1.500	E	F	g
7.000	- 6.444	O	J	j
7.020	- 6.660	E	F	h
7.000	-14.000	O	L	l
7.000	-14.010	E	F	g
7.000	-23.075	O	J	j
7.000	-23.075	E	F	h
8.500	34.389	E	F	g
8.000	30.367	O	I	p
8.500	23.191	E	F	h
8.000	19.876	O	P	i
8.497	13.590	E	F	g
8.000	10.949	O	I	p
8.463	5.885	E	F	h
8.000	3.441	O	P	i
8.275	0.535	E	F	g
8.000	- 3.441	O	I	p
8.073	- 4.170	E	F	h
8.000	-10.949	O	P	i
8.005	-11.015	E	F	g
8.000	-19.876	O	I	p
8.000	-19.879	E	F	h
8.000	-30.367	O	P	i
8.000	-30.367	E	F	g

Table 1 - (continued)

ν	μ	Parity	$\Theta(\theta)$	$\Phi(\varphi)$
9.500	-	E	F	g
9.000	38.660	O	J	j
9.500	-	E	F	h
9.000	26.751	O	L	l
~ 9.500	-	E	F	g
9.000	16.431	O	J	j
-	-	E	F	h
9.000	7.640	O	L	l
-	-	E	F	g
9.000	0.000	O	J	j
-	-	E	F	h
9.000	-7.640	O	L	l
-	-	E	F	g
9.000	-16.413	O	J	j
~ 9.000	~ -16.4	E	F	h
9.000	-26.751	O	L	l
9.000	-26.751	E	F	g
9.000	-38.660	O	J	j
9.000	-38.660	E	F	h

Table 2 - Dirchlet Eigenfunction Coefficients

E ν=0.296 μ=0.090	
A_0 = 1.048	B_0 = 1.036
A_1 =-0.057	B_2 =-0.034
A_{-1}= 0.011	B_4 =-0.002
A_2 =-0.003	
A_{-2}= 0.001	

E ν= 1.425 μ=0.915	
A_0 = 0.173	B_0 = 1.417
A_1 = 0.845	B_2 =-0.405
A_{-1}=-0.021	B_4 =-0.011
A_2 = 0.004	B_6 =-0.001
A_{-2}=-0.001	

O ν=1.000 μ=0.000	
A_1 = 1.000	B_1 = 1.000
A_m = 0,m>1	B_m = 0,m>1

E ν=1.130 μ=-0.455	
A_0 = 1.372	B_1 = 1.010
A_1 =-0.299	B_3 =-0.009
A_{-1}=-0.064	B_5 =-0.001
A_2 =-0.006	
A_{-2}=-0.003	

E ν=2.480 μ=2.670	
A_0 = 0.121	B_0 = 2.470
A_1 = 0.043	B_2 =-0.015
A_{-1}= 0.842	B_4 = 0.040
A_2 =-0.006	B_6 = 0.002
A_{-2}= 0.001	

O ν=2.000 μ=1.500	
A_2 = 0.500	B_1 = 1.414
A_m = 0,m>2	B_m = 0,m>1

E ν=2.290 μ=0.215	
A_0 = 0.343	B_1 = 1.166
A_1 = 0.831	B_3 =-0.160
A_{-1}=-0.113	B_5 =-0.005
A_2 =-0.056	
A_{-2}=-0.001	
A_3 =-0.003	

Table 2 - (continued)

O $\nu=2.000$ $\mu=-1.500$	E $\nu=2.040$ $\mu=-1.705$
$A_1 = 1.414$ $B_2 = 0.500$ $A_m = 0, m>1$ $B_m = 0, m>2$	$A_0 = 2.297$ $B_0 = 1.409$ $A_1 = -1.000$ $B_2 = 0.862$ $A_{-1} = -0.325$ $B_4 = -0.003$ $A_2 = 0.033$ $A_{-2} = -0.007$ $A_3 = 0.002$ $A_{-3} = -0.001$

E $\nu= 3.495$ $\mu= 5.440$	O $\nu=3.000$ $\mu=3.873$	E $\nu=3.410$ $\mu=1.535$
$A_0 = 0.023$ $B_0 = 4.944$ $A_1 = 0.104$ $B_2 = -4.343$ $A_{-1} = 0.009$ $B_4 = 0.391$ $A_2 = 0.865$ $B_6 = 0.007$ $A_{-2} = -0.001$ $B_8 = 0.001$	$A_1 = 0.032$ $B_1 = 1.968$ $A_3 = 0.323$ $B_3 = -0.323$ $A_m = 0, m>3$ $B_m = 0, m>3$	$A_0 = 0.248$ $B_1 = 1.539$ $A_1 = 0.122$ $B_3 = -0.552$ $A_{-1} = 0.737$ $B_5 = 0.013$ $A_2 = -0.044$ $B_7 = 0.001$ $A_{-2} = -0.059$ $A_{-3} = -0.003$
O $\nu=3.000$ $\mu=0.000$	E $\nu=3.145$ $\mu=-0.825$	O $\nu=3.000$ $\mu=-3.873$
$A_2 = 0.707$ $B_2 = 0.707$ $A_m = 0, m>2$ $B_m = 0, m>2$	$A_0 = 0.503$ $B_0 = 0.276$ $A_1 = 0.997$ $B_2 = 0.820$ $A_{-1} = -0.341$ $B_4 = -0.092$ $A_2 = -0.172$ $B_6 = -0.004$ $A_{-2} = 0.014$ $A_3 = -0.003$ $A_{-3} = 0.001$	$A_1 = 1.968$ $B_1 = 0.032$ $A_3 = -0.323$ $B_3 = 0.323$ $A_m = 0, m>3$ $B_m = 0, m>3$

Table 2 - (continued)

E ν= 3.010 μ=-3.940	
A_0 = 4.575	B_1 = 0.141
A_1 =-2.744	P = 0.860
A_{-1}=-1.152	B_5 =-0.001
A_2 = 0.283	
A_{-2}= 0.030	
A_3 = 0.006	
A_{-3}= 0.001	

E ν=4.499 μ=9.225	O ν=4.000 μ=7.190	E ν=4.470 μ=3.790
A_0 = 0.020 B_0= 10.544 A_1 = 0.005 B_2=-11.260 A_{-1}= 0.098 B_4= 1.752 A_2 = 0.002 B_6=- 0.035 A_{-2}= 0.876 B_8=- 0.001	A_2 = 0.026 B_1 = 3.421 A_4 = 0.237 B_3 =-0.669 A_m = 0,m>4 B_m = 0,m>3	A_0 = 0.077 B_1 = 2.377 A_1 = 0.208 B_3 =-1.525 A_{-1}= 0.041 B_5 = 0.145 A_2 = 0.757 B_7 = 0.003 A_{-2}=-0.015 A_3 = 0.065 A_4 =-0.003
O ν=4.000 μ=2.190	E ν=4.280 μ=0.335	O ν=4.000 μ=-2.190
A_1 = 0.109 B_2 = 0.806 A_3 = 0.435 B_4 =-0.153 A_m = 0,m>3 B_m = 0,m>4	A_0 = 0.374 B_0 = 0.410 A_1 = 0.192 B_2 = 0.849 A_{-1}= 0.701 B_4 =-0.263 A_2 =-0.140 B_6 = 0.004 A_{-2}=-0.132 A_3 = 0.007 A_{-3}=-0.001	A_2 = 0.806 B_1 = 0.109 A_4 =-0.153 B_3 = 0.435 A_m = 0,m>4 B_m = 0,m>3

Table 2 - (continued)

E $\nu=4.050$ $\mu=-2.575$	O $\nu=4.000$ $\mu=-7.190$	E $\nu=4.005$ $\mu=-7.205$
$A_0 = 0.751$ $B_1 = 0.313$ $A_1 = 1.482$ $B_3 = 0.764$ $A_{-1} = -0.889$ $B_5 = -0.074$ $A_2 = -0.464$ $B_7 = -0.003$ $A_{-2} = 0.103$ $A_3 = 0.014$ $A_{-3} = 0.002$ $A_4 = 0.001$	$A_1 = 3.421$ $B_2 = 0.026$ $A_3 = -0.669$ $B_4 = 0.237$ $A_m = 0, m>3$ $B_m = 0, m>4$	$A_0 = 0.001$ $B_0 = 0.023$ $A_1 = -7.029$ $B_2 = 0.107$ $A_{-1} = -3.546$ $B_4 = 0.871$ $A_2 = 1.257$ $A_{-2} = 0.289$ $A_3 = -0.027$ $A_{-3} = 0.005$ $A_4 = -0.001$

E $\nu=5.500$ $\mu=14.013$	O $\nu=5.000$ $\mu=11.489$	E $\nu=5.490$ $\mu=7.110$
$A_0 = 0.006$ $B_0 = 23.021$ $A_1 = 0.016$ $B_2 = -27.689$ $A_{-1} = 0.002$ $B_4 = 6.046$ $A_2 = 0.093$ $B_6 = -0.372$ $A_{-2} = 0.001$ $B_8 = -0.006$ $A_3 = 0.883$ $A_m = 0, m>3$	$A_1 = 0.003$ $B_1 = 5.809$ $A_3 = 0.020$ $B_3 = -1.915$ $A_5 = 0.188$ $B_5 = 0.187$ $A_m = 0, m>5$ $B_m = 0, m>5$	$A_0 = 0.058$ $B_1 = 4.102$ $A_1 = 0.021$ $B_3 = -3.739$ $A_{-1} = 0.195$ $B_5 = 0.651$ $A_2 = 0.011$ $B_7 = -0.013$ $A_{-2} = 0.792$ $B_9 = -0.001$ $A_3 = -0.004$ $A_{-3} = -0.069$ $A_{-4} = -0.003$
O $\nu=5.000$ $\mu=5.196$	E $\nu=5.499$ $\mu=2.110$	O $\nu=5.000$ $\mu=0.000$
$A_2 = 0.095$ $B_2 = 1.319$ $A_4 = 0.306$ $B_4 = -0.306$ $A_m = 0, m>4$ $B_m = 0, m>4$	$A_0 = 0.061$ $B_0 = 0.658$ $A_1 = 0.379$ $B_2 = 0.992$ $A_{-1} = 0.032$ $B_4 = -0.722$ $A_2 = 0.704$ $B_6 = 0.071$ $A_{-2} = -0.026$ $B_8 = 0.001$ $A_3 = -0.151$ $A_{-3} = 0.002$	$A_1 = 0.125$ $B_1 = 0.125$ $A_3 = 0.437$ $B_3 = 0.437$ $A_5 = -0.087$ $B_5 = -0.087$ $A_m = 0, m>5$ $B_m = 0, m>5$

Table 2 - (continued)

E $\nu=5.150$ $\mu=-1.170$		O $\nu=5.000$ $\mu=-5.196$		E $\nu=5.015$ $\mu=-5.335$	
$A_0 = 0.551$ $A_1 = 0.245$ $A_{-1} = 0.778$ $A_2 = -0.366$ $A_{-2} = -0.264$ $A_3 = 0.046$ $A_{-3} = 0.009$ $A_4 = 0.001$	$B_1 = 0.494$ $B_3 = 0.676$ $B_5 = -0.172$ $B_7 = 0.001$	$A_2 = 1.319$ $A_4 = -0.306$ $A_m = 0, m>4$	$B_2 = 9.472$ $B_4 = 0.306$ $B_m = 0, m>4$	$A_0 = 1.220$ $A_1 = 2.615$ $A_{-1} = -2.127$ $A_2 = -1.247$ $A_{-2} = 0.438$ $A_3 = 0.110$ $A_{-3} = -0.010$ $A_4 = 0.002$	$B_0 = 0.062$ $B_2 = 0.228$ $B_4 = 0.785$ $B_6 = -0.071$ $B_8 = -0.003$

O $\nu=5.000$ $\mu=-11.489$		E $\nu=5.000$ $\mu=-11.493$	
$A_1 = 5.809$ $A_3 = -1.915$ $A_5 = 0.187$ $A_m = 0, m>5$	$B_1 = 0.003$ $B_3 = 0.020$ $B_5 = 0.188$ $B_m = 0, m>5$	$A_0 = 21.014$ $A_1 = -15.824$ $A_{-1} = -9.110$ $A_2 = 3.952$ $A_{-2} = 1.261$ $A_3 = -0.264$ $A_{-3} = -0.023$ $A_4 = -0.004$ $A_{-4} = -0.001$	$B_1 = 0.020$ $B_3 = 0.096$ $B_5 = 0.884$

E $\nu=6.500$ $\mu=19.805$		O $\nu=6.000$ $\mu=16.783$		E $\nu=6.499$ $\mu=11.455$	
$A_0 = 0.005$ $A_1 = 0.001$ $A_{-1} = 0.015$ $A_{-2} = 0.091$ $A_{-3} = 0.893$ $A_m = 0, m<-3$	$B_0 = 44.205$ $B_2 = -57.427$ $B_4 = 15.840$ $B_6 = -1.644$ $B_8 = 0.025$ $B_{10} = 0.001$	$A_2 = 0.002$ $A_4 = 0.016$ $A_6 = 0.155$ $A_m = 0, m>6$	$B_1 = 11.141$ $B_3 = -3.973$ $B_5 = 0.438$ $B_n = 0, m>5$	$A_0 = 0.136$ $A_1 = 0.040$ $A_{-1} = 0.050$ $A_2 = 0.153$ $A_{-2} = 0.028$ $A_3 = 0.665$ $A_{-3} = -0.010$ $A_4 = -0.059$ $A_5 = -0.003$	$B_1 = 7.730$ $B_3 = -8.840$ $B_5 = 2.259$ $B_7 = -0.147$ $B_9 = -0.002$

Table 2 - (continued)

O $\nu=6.000$ $\mu=9.114$	E $\nu=6.465$ $\mu=4.850$	O $\nu=6.000$ $\mu=2.832$
$A_1 = 0.015$ $B_2 = 1.813$ $A_3 = 0.074$ $B_4 = -0.780$ $A_5 = 0.235$ $B_6 = 0.083$ $A_m = 0, m>5$ $B_m = 0, m>6$	$A_0 = -0.137$ $B_0 = 0.995$ $A_1 = -0.080$ $B_2 = 1.360$ $A_{-1} = 0.419$ $B_4 = -1.641$ $A_2 = -0.045$ $B_6 = 0.293$ $A_{-2} = 1.016$ $B_8 = -0.006$ $A_3 = 0.032$ $A_{-3} = -0.202$ $A_4 = -0.002$	$A_2 = 0.103$ $B_1 = 0.268$ $A_4 = 0.284$ $B_3 = 0.652$ $A_6 = -0.057$ $B_5 = -0.162$ $A_m = 0, m>6$ $B_m = 0, m>5$
E $\nu=6.282$ $\mu=0.445$	O $\nu=6.000$ $\mu=-2.832$	E $\nu=6.055$ $\mu=-3.380$
$A_0 = -0.411$ $B_1 = 0.688$ $A_1 = 0.688$ $B_3 = 0.629$ $A_{-1} = -0.189$ $B_5 = -0.343$ $A_2 = 1.002$ $B_7 = 0.026$ $A_{-2} = 2.790$ $B_9 = 0.001$ $A_3 = -0.347$ $A_{-3} = -0.037$ $A_4 = 0.014$ $A_5 = 0.001$	$A_1 = 0.268$ $B_2 = 0.103$ $A_3 = 0.652$ $B_4 = 0.284$ $A_5 = -0.162$ $B_6 = -0.057$ $A_m = 0, m>5$ $B_m = 0, m>6$	$A_0 = 1.056$ $B_0 = 0.028$ $A_1 = 0.396$ $B_2 = 0.391$ $A_{-1} = 1.256$ $B_4 = 0.745$ $A_2 = -1.384$ $B_6 = -0.163$ $A_{-2} = -0.683$ $A_3 = 0.303$ $A_{-3} = 0.063$ $A_4 = -0.008$ $A_{-4} = 0.001$
O $\nu=6.000$ $\mu=-9.114$	E $\nu=6.005$ $\mu=-9.160$	O $\nu=6.000$ $\mu=-16.783$
$A_2 = 1.813$ $B_1 = 0.015$ $A_4 = -0.780$ $B_3 = 0.074$ $A_6 = 0.082$ $B_5 = 0.235$ $A_m = 0, m>6$ $B_m = 0, m>5$	$A_0 = 1.781$ $B_1 = 0.069$ $A_1 = 4.462$ $B_3 = 0.199$ $A_{-1} = -3.923$ $B_5 = 0.808$ $A_2 = -2.850$ $B_7 = -0.072$ $A_{-2} = 1.186$ $B_9 = -0.003$ $A_3 = 0.439$ $A_{-3} = -0.085$ $A_4 = -0.009$ $A_{-4} = -0.001$	$A_1 = 11.141$ $B_2 = 0.002$ $A_3 = -3.973$ $B_4 = 0.016$ $A_5 = 0.438$ $B_6 = 0.155$ $A_m = 0, m>5$ $B_m = 0, m>6$

Table 2 - (continued)

E $\nu=6.000$ $\mu=-16.783$	
A_0 =40.690	B_0 = 0.003
A_1=-32.045	B_2 = 0.015
A_{-1}=-20.297	B_4 = 0.092
A_2 =10.063	B_6 = 0.890
A_{-2}= 3.937	
A_3 =-1.139	
A_{-3}=-0.224	
A_4 = 0.018	
A_{-4}=-0.003	
A_5 = 0.001	

E $\nu=7.500$ $\mu=26.597$	
A_0 = 0.007	B_0=-55.346
A_1 = 0.004	B_2=-75.696
A_{-1}= 0.002	B_4= 24.784
A_2 = 0.014	B_6=- 3.603
A_3 = 0.088	B_8= 0.167
A_4 = 0.886	B_{10}= 0.002

O $\nu=7.000$ $\mu=23.075$	
A_3 = 0.004	B_1= 14.115
A_5 = 0.023	B_3=- 6.380
A_7 = 0.124	B_5= 1.264
A_m = 0,m>7	B_7=- 0.042
	B_m= 0,m>7

E $\nu=7.500$ $\mu=16.820$	
A_0 = 0.124	B_1= 13.374
A_1 = 0.035	B_3=-17.621
A_{-1}= 0.036	B_5= 5.898
A_2 = 0.012	B_7=- 0.661
A_{-2}= 0.154	B_9= 0.010
A_3 = 0.007	
A_{-3}= 0.700	
A_4 =-0.003	
A_{-4}=-0.062	
A_{-5}=-0.003	
A_m = 0,m>4	

Table 2 - (continued)

0 $\nu=7.000$ $\mu=14.000$	E $\nu=7.492$ $\mu=8.695$	0 $\nu=7.000$ $\mu=6.444$
$A_2 = 0.015$ $B_2 = 3.314$ $A_4 = 0.059$ $B_4 =-1.591$ $A_6 = 0.191$ $B_6 = 0.191$ $A_m = 0, m>6$ $B_m = 0, m>6$	$A_0 =-2.035$ $B_0 = 2.647$ $A_1 =-0.759$ $B_2 = 3.500$ $A_{-1}=-1.136$ $B_4 =-6.936$ $A_2 = 1.665$ $B_6 = 1.917$ $A_{-2}=-0.660$ $B_8 =-0.127$ $A_3 = 4.368$ $B_{10}=-0.002$ $A_{-3}= 0.453$ $A_4 =-0.865$ $A_{-4}=-0.029$ $A_{-5}=-0.001$	$A_1 = 0.018$ $B_1 = 0.338$ $A_3 = 0.077$ $B_3 = 0.747$ $A_5 = 0.208$ $B_5 =-0.374$ $A_7 =-0.042$ $B_7 = 0.041$ $A_m = 0, m>7$ $B_m = 0, m>7$
E $\nu=7.393$ $\mu=2.660$	0 $\nu=7.000$ $\mu=0.000$	E $\nu=7.157$ $\mu=-1.500$
$A_0 = 0.343$ $B_1 = 0.970$ $A_1 = 1.527$ $B_3 = 0.644$ $A_{-1}=-0.358$ $B_5 =-0.736$ $A_2 = 0.758$ $B_7 = 0.123$ $A_{-2}=-0.578$ $B_9 =-0.002$ $A_3 =-1.024$ $A_{-3}= 0.199$ $A_4 = 0.140$ $A_{-4}=-0.009$ $A_5 = 0.001$	$A_2 = 0.221$ $B_2 = 0.221$ $A_4 = 0.389$ $B_4 = 0.389$ $A_6 =-0.097$ $B_6 =-0.097$ $A_m = 0, m>6$ $B_m = 0, m>6$	$A_0 =-1.751$ $B_0 =-0.082$ $A_1 = 0.852$ $B_2 = 0.644$ $A_{-1}=-0.615$ $B_4 = 0.745$ $A_2 = 0.936$ $B_6 =-0.329$ $A_{-2}= 2.628$ $B_8 = 0.021$ $A_3 =-0.527$ $B_{10}= 0.001$ $A_{-3}=-0.594$ $A_4 = 0.051$ $A_{-4}= 0.017$ $A_5 = 0.001$ $A_{-5}= 0.001$
0 $\nu=7.000$ $\mu=-6.444$	E $\nu=7.020$ $\mu=-6.660$	0 $\nu=7.000$ $\mu=-14.000$
$A_1 = 0.338$ $B_1 = 0.018$ $A_3 = 0.747$ $B_3 = 0.077$ $A_5 =-0.374$ $B_5 = 0.208$ $A_7 = 0.041$ $B_7 =-0.042$ $A_m = 0, m>7$ $B_m = 0, m>7$	$A_0 =-0.504$ $B_1 =-0.086$ $A_1 =-0.194$ $B_3 = 0.378$ $A_{-1}=-0.547$ $B_5 = 0.890$ $A_2 = 2.708$ $B_7 =-0.183$ $A_{-2}= 0.442$ $A_3 =-0.903$ $A_{-3}=-0.072$ $A_4 = 0.067$ $A_{-4}= 0.001$ $A_5 = 0.001$	$A_2 = 3.314$ $B_2 = 0.015$ $A_4 =-1.591$ $B_4 = 0.059$ $A_6 = 0.191$ $B_6 = 0.191$ $A_m = 0, m>6$ $B_m = 0, m>6$

Table 2 - (continued)

E ν=7.000 μ=-14.010		O ν=7.000 μ=-23.075		E ν=7.000 μ=-23.075	
A_0 = 1.580	B_0 = 0.075	A_1 =14.115	B_3 = 0.004	A_0 =51.657	B_1 = 0.003
A_1 = 5.199	B_2 = 0.044	A_3 =-6.380	B_5 = 0.023	A_1=-41.656	B_3 = 0.014
A_{-1}=-3.923	B_4 = 0.175	A_5 = 1.264	B_7 = 0.124	A_{-1}=-28.427	B_5 = 0.090
A_2 =-4.075	B_6 = 0.778	A_7 =-0.042	B_m = 0,m>7	A_2 =15.419	B_7 = 0.894
A_{-2}= 1.556	B_8 =-0.069	A_m = 0,m>7		A_{-2}= 6.990	
A_3 = 0.905	B_{10}=-0.003			A_3 =-2.446	
A_{-3}=-0.192				A_{-3}=-0.670	
A_4 =-0.054				A_4 = 0.121	
A_{-4}= 0.003				A_{-4}= 0.010	
A_5 =-0.001				A_5 = 0.002	

E ν=8.500 μ=34.389		O ν=8.000 μ=30.367		E ν=8.500 μ=23.191	
A_{-1}= 0.002	B_0 =44.092	A_4 = 0.002	B_1 =33.142	A_0 = 0.391	B_1 =15.874
A_{-2}= 0.013	B_2=-62.390	A_6 = 0.011	B_3=-15.594	A_1 = 0.101	B_3=-22.791
A_{-3}= 0.087	B_4 =23.282	A_8 = 0.116	B_5 = 3.291	A_{-1}= 0.109	B_5 = 9.335
A_{-4}= 0.898	B_6 =-4.316	A_m = 0,m>8	B_7 =-0.200	A_2 = 0.030	B_7 =-1.492
A_m =0,m<-4	B_8 = 0.336		B_m = 0,m>7	A_{-2}= 0.039	B_9 = 0.072
	B_{10}=-0.004			A_3 = 0.134	B_{11}= 0.001
				A_{-3}= 0.022	
				A_4 = 0.632	
				A_{-4}=-0.008	
				A_5 =-0.056	
				A_6 =-0.002	
				A_m =0,m<-4	

Table 2 - (continued)

O $\nu=8.000$ $\mu=19.876$	E $\nu=8.497$ $\mu=13.590$	O $\nu=8.000$ $\mu=10.949$
$A_1 = 0.002$ $B_2 = 4.614$ $A_3 = 0.012$ $B_4 = -2.999$ $A_5 = 0.049$ $B_6 = 0.678$ A_7 [illegible]2 $B_8 = -0.037$ $A_m = 0, m>7$ $B_m = 0, m>8$	$A_0 = -5.594$ $B_0 = -2.223$ $A_1 = -1.931$ $B_2 = -3.223$ $A_{-1} = -2.044$ $B_4 = 9.701$ $A_2 = -1.042$ $B_6 = -3.670$ $A_{-2} = 3.634$ $B_8 = 0.428$ $A_3 = -0.616$ $B_{10} = -0.007$ $A_{-3} = 10.204$ $A_4 = 0.411$ $A_{-4} = -1.995$ $A_5 = -0.026$ $A_6 = -0.001$	$A_2 = 0.017$ $B_1 = 0.705$ $A_4 = 0.059$ $B_3 = 1.233$ $A_6 = 0.165$ $B_5 = -0.726$ $A_8 = -0.032$ $B_7 = 0.092$ $A_m = 0, m>8$ $B_m = 0, m>7$
E $\nu=8.463$ $\mu=5.885$	O $\nu=8.000$ $\mu=3.441$	E $\nu=8.275$ $\mu=0.535$
$A_0 = 0.378$ $B_1 = 1.957$ $A_1 = 0.170$ $B_3 = 1.051$ $A_{-1} = 0.764$ $B_5 = -2.712$ $A_2 = -0.127$ $B_7 = 0.753$ $A_{-2} = 0.400$ $B_9 = -0.049$ $A_3 = -0.224$ $B_{11} = -0.001$ $A_{-3} = -0.503$ $A_4 = 0.076$ $A_{-4} = 0.070$ $A_5 = -0.004$	$A_1 = 0.048$ $B_2 = 0.251$ $A_3 = 0.163$ $B_4 = 0.386$ $A_5 = 0.267$ $B_6 = -0.206$ $A_7 = -0.065$ $B_8 = 0.023$ $A_m = 0, m>7$ $B_m = 0, m>8$	$A_0 = 2.827$ $B_0 = -26.490$ $A_1 = 0.573$ $B_2 = 23.183$ $A_{-1} = -1.267$ $B_4 = 17.984$ $A_2 = 0.221$ $B_6 = -16.021$ $A_{-2} = -1.479$ $B_8 = 2.370$ $A_3 = -0.776$ $B_{10} = -0.025$ $A_{-3} = 0.815$ $B_{12} = -0.001$ $A_4 = 0.177$ $A_{-4} = -0.083$ $A_5 = -0.006$ $A_{-5} = -0.001$
O $\nu=8.000$ $\mu=-3.441$	E $\nu=8.073$ $\mu=-4.170$	O $\nu=8.000$ $\mu=-10.949$
$A_2 = 0.251$ $B_1 = 0.048$ $A_4 = 0.386$ $B_3 = 0.163$ $A_6 = -0.206$ $B_5 = 0.267$ $A_8 = 0.023$ $B_7 = -0.065$ $A_m = 0, m>8$ $B_m = 0, m>7$	$A_0 = 0.245$ $B_1 = -0.660$ $A_1 = -0.005$ $B_3 = 0.854$ $A_{-1} = 0.071$ $B_5 = 1.254$ $A_2 = -0.004$ $B_7 = -0.476$ $A_{-2} = 1.014$ $B_9 = 0.027$ $A_3 = 0.004$ $B_{11} = 0.001$ $A_{-3} = -0.352$ $A_4 = -0.001$ $A_{-4} = 0.027$	$A_1 = 0.705$ $B_2 = 0.017$ $A_3 = 1.233$ $B_4 = 0.059$ $A_5 = -0.726$ $B_6 = 0.165$ $A_7 = 0.092$ $B_8 = -0.032$ $A_m = 0, m>7$ $B_m = 0, m>8$

Table 2 - (continued)

E ν=8.005 μ=-11.015		0 ν=8.000 μ=-19.876		E ν=8.000 μ=-19.879	
A_0 = 0.136 A_1 = 0.063 A_{-1}= 0.146 A_2 = 1.306 A_{-2}=-0.163 A_3 =-0.601 A_{-3}= 0.039 A_4 = 0.078 A_{-4}=-0.002 A_5 =-0.001	B_0 =-0.312 B_2 =-0.230 B_4 = 0.491 B_6 = 1.312 B_8 =-0.260	A_2 = 4.614 A_4 =-2.999 A_6 = 0.678 A_8 =-0.037 A_m = 0,m>8	B_1 = 0.002 B_3 = 0.012 B_5 = 0.049 B_7 = 0.162 B_m = 0,m>7	A_0 = 0.847 A_1 = 4.422 A_{-1}=-2.183 A_2 =-4.015 A_{-2}= 1.061 A_3 = 1.163 A_{-3}=-0.187 A_4 =-0.120 A_{-4}= 0.010 A_5 = 0.002	B_1 = 0.127 B_3 = 0.037 B_5 = 0.161 B_7 = 0.744 B_9 =-0.066 B_{11}=-0.003

0 ν=8.000 μ=-30.367		E ν=8.000 μ=-30.367	
A_1 =33.142 A_3=-15.594 A_5 = 3.291 A_7 =-0.200 A_m = 0,m>7	B_4 = 0.002 B_6 = 0.011 B_8 = 0.116 B_m = 0,m>8	A_0 =42.222 A_1=-34.393 A_{-1}=-24.936 A_2=14.399 A_{-2}= 7.316 A_3 =-2.908 A_{-3}=-0.982 A_4 = 0.241 A_{-4}= 0.043 A_5 =-0.003 A_{-5}= 0.001	B_2 = 0.002 B_4 = 0.013 B_6 = 0.088 B_8 = 0.896

0 ν=9.000 μ=38.660		0 ν=9.000 μ=26.751		0 ν=9.000 μ=16.413	
A_5 = 0.001 A_7 = 0.010 A_9 = 0.103 A_m = 0,m>9	B_1 =-11.517 B_3 = 6.208 B_5 =-1.776 B_7 = 0.247 B_9 = 0.116 B_m = 0,m>9	A_2 = 0.002 A_4 = 0.010 A_6 = 0.041 A_8 = 0.140 A_m = 0,m>8	B_2 =10.015 B_4 =-6.916 B_6 = 1.685 B_8 =-0.133 B_m = 0,m>8	A_1 = 0.003 A_3 = 0.014 A_5 = 0.047 A_7 = 0.137 A_9 =-0.027 A_m = 0,m>9	B_1 = 1.021 B_3 = 1.612 B_5 =-1.405 B_7 = 0.346 B_9 =-0.028 B_m = 0,m>9

Table 2 - (continued)

0 $\nu=9.000$ $\mu=7.640$	0 $\nu=9.000$ $\mu=0.000$	0 $\nu=9.000$ $\mu=-7.640$
$A_2 = 0.048$ $B_2 = 0.503$ $A_4 = 0.123$ $B_4 = 0.566$ $A_6 = 0.202$ $B_6 = -0.375$ $A_8 = -0.048$ $B_8 = 0.048$ $A_m = 0, m>8$ $B_m = 0, m>8$	$A_1 = 0.055$ $B_1 = 0.055$ $A_3 = 0.172$ $B_3 = 0.172$ $A_5 = 0.241$ $B_5 = 0.241$ $A_7 = -0.129$ $B_7 = -0.129$ $A_9 = 0.014$ $B_9 = 0.014$ $A_m = 0, m>9$ $B_m = 0, m>9$	$A_2 = 0.503$ $B_2 = 0.048$ $A_4 = 0.568$ $B_4 = 0.123$ $A_6 = -0.375$ $B_6 = 0.202$ $A_8 = 0.048$ $B_8 = -0.048$ $A_m = 0, m>8$ $B_m = 0, m>8$
0 $\nu=9.000$ $\mu=-16.413$	0 $\nu=9.000$ $\mu=-26.751$	0 $\nu=9.000$ $\mu=-38.660$
$A_1 = 1.021$ $B_1 = 0.003$ $A_3 = 1.612$ $B_3 = 0.014$ $A_5 = -1.405$ $B_5 = 0.047$ $A_7 = 0.346$ $B_7 = 0.137$ $A_9 = -0.028$ $B_9 = -0.027$ $A_m = 0, m>9$ $B_m = 0, m>9$	$A_2 = 10.015$ $B_2 = 0.002$ $A_4 = -6.916$ $B_4 = 0.010$ $A_6 = 1.685$ $B_6 = 0.041$ $A_8 = -0.133$ $B_8 = 0.140$ $A_m = 0, m>8$ $B_m = 0, m>8$	$A_1 = -11.517$ $B_5 = 0.001$ $A_3 = 6.208$ $B_7 = 0.010$ $A_5 = -1.776$ $B_9 = 0.103$ $A_7 = 0.247$ $B_m = 0, m>9$ $A_9 = 0.116$ $A_m = 0, m>9$
	E $\nu=9.000$ $\mu=-38.660$	
	$A_0 = 30.271$ $B_3 = 0.002$ $A_1 = -24.690$ $B_5 = 0.013$ $A_{-1} = -18.818$ $B_7 = 0.087$ $A_2 = 11.362$ $B_9 = 0.898$ $A_{-2} = 6.329$ $B_m = 0, m>9$ $A_3 = -2.757$ $A_{-3} = -1.084$ $A_4 = 0.321$ $A_{-4} = 0.079$ $A_5 = -0.013$ $A_{-5} = -0.001$	

Table 3 - Eigenvalues of the Neumann Problem and the Form of the Angular Eigenfunctions

$F = \sum A_m \sin (2m - 1/2)\ \theta$

$G = \sum A_{2m} \cos 2m\ \theta$

$H = \sum A_{2m+1} \cos (2m + 1)\ \theta$

$L = \sqrt{1-k^2 \cos^2 \theta} \sum A_{2m} \cos 2m\ \theta$

$P = \sqrt{1-k^2 \cos^2 \theta} \sum A_{2m+1} \cos (2m + 1)\ \theta$

$g = \sum B_{2m} \cos 2m\ \varphi$

$h = \sum B_{2m+1} \cos (2m + 1)\ \varphi$

$i = \sum B_{2m} \sin 2m\ \varphi$

$j = \sum B_{2m+1} \sin (2m + 1)\ \varphi$

$l = \sqrt{1-k'^2 \cos^2 \varphi} \sum B_{2m} \cos 2m\ \varphi$

$p = \sqrt{1-k'^2 \cos^2 \varphi} \sum B_{2m+1} \cos (2m + 1)\ \varphi$

ν	μ	Parity	$\Theta(\theta)$	$\Phi(\varphi)$
0.000	0.000	E	G	g
1.000	0.500	E	H	l
0.814	-0.194	O	F	j
1.000	-0.500	E	L	h
2.000	1.732	E	G	g
1.595	0.795	O	F	j
2.000	0.000	E	P	p
1.955	-1.552	O	F	i
2.000	-1.732	F	G	g
3.000	3.950	E	H	l
2.520	2.625	O	F	j
3.000	0.950	E	L	h
2.803	·0.350	O	F	i
3.000	-0.950	E	H	l
2.990	-3.890	O	F	j
3.000	-3.950	E	L	h
4.000	7.211	E	G	g
3.505	5.430	O	F	j

Table 3 - (continued)

ν	μ	Parity	$\Theta(\theta)$	$\Phi(\varphi)$
4.000	2.646	E	P	p
3.617	1.260	O	F	i
4.000	0	E	G	g
3.940	- 2.315	O	F	j
4.000	- 2.646	E	P	p
3.998	- 7.195	O	F	i
4.000	- 7.211	E	G	g
5.000	11.494	E	H	l
4.500	9.218	O	F	j
5.000	5.363	E	L	h
4.530	3.680	O	F	i
5.000	1.369	E	H	l
4.795	- 0.495	O	F	j
5.000	- 1.369	E	L	h
4.984	- 5.230	O	F	i
5.000	- 5.363	E	H	l
5.000	-11.491	O	F	j
5.000	-11.494	E	L	h
6.000	16.784	E	G	g
5.500	14.012	O	F	j
6.000	9.165	E	P	p
5.507	7.060	O	F	i
6.000	3.507	E	G	g
5.627	1.692	O	F	j
6.000	0.000	E	P	p
5.927	- 3.015	O	F	i
6.000	- 3.507	E	C	g
5.997	- 9.120	O	F	j
6.000	- 9.165	E	P	p
6.000	-16.783	O	F	i
6.000	-16.784	E	G	g

Table 3 - (continued)

ν	μ	Parity	$\Theta(\theta)$	$\Phi(\varphi)$
7.000	23.076	E	H	l
6.500	19.804	O	F	j
7.000	14.014	E	L	h
6.501	11.440	O	F	i
7.000	6.708	E	H	l
6.540	4.675	O	F	j
7.000	1.770	E	L	h
6.792	- 0.630	O	F	i
7.000	- 1.770	E	H	l
6.980	- 6.500	O	F	j
7.000	- 6.708	E	L	h
7.000	-14.002	O	F	i
7.000	-14.014	E	H	l
7.000	-23.075	O	F	j
7.000	-23.076	E	L	h
8.000	30.368	E	G	g
7.500	26.597	O	F	j
8.000	19.880	E	P	p
7.500	16.816	O	F	i
8.000	11.036	E	G	g
7.510	8.625	O	F	j
8.000	4.334	E	P	p
7.632	2.095	O	F	i
8.000	0.000	E	G	g
7.923	- 3.685	O	F	j
8.000	- 4.334	E	P	p
7.995	-10.965	O	F	i
8.000	-11.036	E	G	g
8.000	-19.877	O	F	j
8.000	-19.880	E	P	p
8.000	-30.368	O	F	i
8.000	-30.368	E	G	g

Table 3 - (continued)

ν	μ	Parity	$\Theta(\theta)$	$\Phi(\varphi)$
9.000	38.660	E	H	l
8.500	34.389	O	F	j
9.000	26.752	E	L	h
8.500	23.190	O	F	i
9.000	16.438	E	H	l
8.503	13.570	O	F	j
9.000	8.004	E	L	h
8.545	5.630	O	F	i
9.000	2.158	E	H	l
8.787	- 0.770	O	F	j
9.000	- 2.158	E	L	h
8.980	- 7.710	O	F	i
9.000	- 8.004	E	H	l
9.000	-16.420	O	F	j
9.000	-16.438	E	L	h
9.000	-26.751	O	F	i
9.000	-26.752	E	H	l
9.000	-38.660	O	F	j
9.000	-38.660	E	L	h

Table 4 - Neumann Eigenfunction Coefficients

E ν=0.000 μ=0.000
A_0 = 1.000 B_0 = 1.000 A_m = 0,m>0 B_m = 0,m>0

E ν=1.000 μ=0.500	0 ν=0.814 μ=-0.194	E ν=1.000 μ=-0.500
A_1 = 1.000 B_0 = 1.414 A_m = 0,m>1 B_m = 0,m>0	A_0 =-1.473 B_1 = 0.964 A_1 = 0.214 B_3 = 0.010 A_{-1}= 0.030 B_5 = 0.001 A_2 = 0.007 A_{-2}= 0.002 A_3 = 0.001	A_0 = 1.414 B_1 = 1.000 A_m = 0,m>0 B_m = 0,m>1

E ν=2.000 μ=1.732	0 ν=1.595 μ=0.795	E ν=2.000 μ=0.000
A_0 = 0.134 B_0 = 1.866 A_2 = 0.866 B_2 =-0.866 A_m = 0,m>2 B_m = 0,m>2	A_0 = 0.174 B_1 = 1.160 A_1 = 0.689 B_3 =-0.048 A_{-1}=-0.026 B_5 =-0.003 A_2 =-0.005 A_{-2}=-0.001	A_1 = 1.414 B_1 = 1.414 A_m = 0,m>1 B_m = 0,m>1

0 ν=1.955 μ=-1.552	E ν=2.000 μ=-1.732
A_0 =-1.360 B_2 = 0.496 A_1 = 0.568 B_4 = 0.002 A_{-1}= 0.179 A_2 =-0.016 A_{-2}= 0.004 A_3 =-0.001	A_0 = 1.866 B_0 = 0.134 A_2 =-0.866 B_2 = 0.866 A_m = 0,m>2 B_m = 0,m>2

Table 4 - (continued)

E $\nu=3.000$ $\mu=3.950$	0 $\nu=2.520$ $\mu=2.625$	E $\nu=3.000$ $\mu=0.950$
$A_1 = 0.138 \quad B_0 = 3.856$ $A_3 = 0.862 \quad B_2 =-2.442$ $A_m = 0,m>2 \quad B_m = 0,m>2$	$A_0 =-0.060 \quad B_1 = 1.601$ $A_1 =-0.022 \quad B_3 =-0.193$ $A_{-1}=-0.397 \quad B_5 =-0.004$ $A_2 = 0.033$ $A_{-2}= 0.001$	$A_0 = 0.389 \quad B_1 = 1.364$ $A_2 = 1.025 \quad B_3 =-0.364$ $A_m = 0,m>2 \quad B_m = 0,m>3$
0 $\nu=2.803$ $\mu=-0.350$	E $\nu=3.000$ $\mu=-0.950$	0 $\nu=2.990$ $\mu=-3.890$
$A_0 = 0.346 \quad B_2 = 0.583$ $A_1 = 0.718 \quad B_4 =-0.038$ $A_{-1}=-0.180 \quad B_6 =-0.002$ $A_2 =-0.091$ $A_{-2}= 0.003$ $A_3 =-0.003$	$A_1 = 1.364 \quad B_0 = 0.389$ $A_3 =-0.364 \quad B_2 = 1.025$ $A_m = 0,m>3 \quad B_m = 0,m>2$	$A_0 =-1.720 \quad B_1 = 0.032$ $A_1 = 1.025 \quad B_3 = 0.322$ $A_{-1}= 0.429$ $A_2 =-0.104$ $A_{-2}=-0.011$ $A_3 =-0.002$ $A_{-3}=-0.001$

E $\nu=3.000$ $\mu=-3.950$
$A_0 = 3.856 \quad B_1 = 0.138$ $A_2 =-2.442 \quad B_3 = 0.862$ $A_m = 0,m>2 \quad B_m = 0,m>3$

E $\nu=4.000$ $\mu=7.211$	0 $\nu=3.505$ $\mu=5.430$	E $\nu=4.000$ $\mu=2.646$
$A_0 = 0.019 \quad B_0 = 7.246$ $A_2 = 0.106 \quad B_2 =-7.120$ $A_4 = 0.875 \quad B_4 = 0.874$ $A_m = 0,m>4 \quad B_m = 0,m>4$	$A_0 = 0.013 \quad B_1 = 2.505$ $A_1 = 0.035 \quad B_3 =-0.524$ $A_{-1}= 0.005 \quad B_5 = 0.012$ $A_2 = 0.275 \quad B_7 = 0.001$ $A_{-2}=-0.001$	$A_1 = 0.479 \quad B_1 = 2.350$ $A_3 = 0.935 \quad B_3 =-0.936$ $A_m = 0,m>3 \quad B_m = 0,m>3$

Table 4 - (continued)

O $\nu=3.617$ $\mu=1.260$	E $\nu=4.000$ $\mu=0.000$	O $\nu=3.940$ $\mu=-2.315$
$A_0=-0.173$ $B_2=0.718$ $A_1=-0.088$ $B_4=-0.106$ $A_{-1}=-0.455$ $B_6=-0.002$ $A_2=0.038$ $A_{-2}=0.047$ $A_{-3}=0.002$	$A_0=0.375$ $B_0=0.375$ $A_2=0.833$ $B_2=0.833$ $A_4=-0.208$ $B_4=-0.208$ $A_m=0, m>4$ $B_m=0, m>4$	$A_0=0.365$ $B_1=0.071$ $A_1=0.714$ $B_3=0.363$ $A_{-1}=-0.409$ $B_5=-0.030$ $A_2=-0.211$ $B_7=-0.001$ $A_{-2}=0.044$ $A_3=0.005$ $A_{-3}=0.001$
E $\nu=4.000$ $\mu=-2.646$	O $\nu=3.998$ $\mu=-7.195$	E $\nu=4.000$ $\mu=-7.211$
$A_1=2.350$ $B_1=0.479$ $A_3=-0.936$ $B_3=0.935$ $A_m=0, m>3$ $B_m=0, m>3$	$A_0=-2.639$ $B_2=0.026$ $A_1=1.836$ $B_4=0.237$ $A_{-1}=0.927$ $A_2=-0.327$ $A_{-2}=-0.075$ $A_3=0.007$ $A_{-3}=-0.001$	$A_0=7.246$ $B_0=0.019$ $A_2=-7.120$ $B_2=0.106$ $A_4=0.874$ $B_4=0.875$ $A_m=0, m>4$ $B_m=0, m>4$

E $\nu=5.000$ $\mu=11.494$	O $\nu=4.500$ $\mu=9.218$	E $\nu=5.000$ $\mu=5.363$
$A_1=0.020$ $B_0=16.709$ $A_3=0.096$ $B_2=-17.796$ $A_5=0.884$ $B_4=2.501$ $A_m=0, m>5$ $B_m=0, m>4$	$A_0=-0.003$ $B_1=4.281$ $A_1=-0.001$ $B_3=-1.250$ $A_{-1}=-0.023$ $B_5=0.092$ $A_{-2}=-0.210$ $B_7=0.002$ $A_m=0, m<-2$	$A_0=0.106$ $B_1=3.133$ $A_2=0.363$ $B_3=-2.468$ $A_4=0.946$ $B_5=0.335$ $A_m=0, m>4$ $B_m=0, m>5$

Table 4 - (continued)

O $\nu=4.530$ $\mu=3.680$	E $\nu=5.000$ $\mu=1.369$	O $\nu=4.795$ $\mu=-0.495$
$A_0 = 0.028$ $A_1 = 0.086$ $A_{-1} = 0.015$ $A_2 = 0.303$ $A_{-2} = -0.006$ $A_3 = -0.028$ $A_4 = -0.001$ $B_2 = 0.969$ $B_4 = -0.245$ $B_6 = 0.006$	$A_1 = 0.471$ $A_3 = 0.685$ $A_5 = -0.156$ $A_m = 0, m>5$ $B_0 = 0.704$ $B_2 = 1.151$ $B_4 = -0.442$ $B_m = 0, m>4$	$A_0 = -0.295$ $A_1 = -0.141$ $A_{-1} = -0.459$ $A_2 = 0.157$ $A_{-2} = 0.125$ $A_3 = -0.015$ $A_{-3} = -0.002$ $B_1 = 0.112$ $B_3 = 0.420$ $B_5 = -0.073$ $B_7 = -0.001$
E $\nu=5.000$ $\mu=-1.369$	O $\nu=4.984$ $\mu=-5.230$	E $\nu=5.000$ $\mu=-5.363$
$A_0 = 0.704$ $A_2 = 1.151$ $A_4 = -0.442$ $A_m = 0, m>4$ $B_1 = 0.471$ $B_3 = 0.685$ $B_5 = -0.156$ $B_m = 0, m>5$	$A_0 = 0.406$ $A_1 = 0.867$ $A_{-1} = -0.702$ $A_2 = -0.409$ $A_{-2} = 0.143$ $A_3 = 0.035$ $A_{-3} = -0.003$ $A_4 = 0.001$ $B_2 = 0.059$ $B_4 = 0.256$ $B_6 = -0.022$ $B_8 = -0.001$	$A_1 = 3.133$ $A_3 = -2.468$ $A_5 = 0.335$ $A_m = 0, m>5$ $B_0 = 0.106$ $B_2 = 0.363$ $B_4 = 0.946$ $B_m = 0, m>4$
O $\nu=5.000$ $\mu=-11.491$	E $\nu=5.000$ $\mu=-11.494$	
$A_0 = -4.535$ $A_1 = 3.417$ $A_{-1} = 1.967$ $A_2 = -0.854$ $A_{-2} = -0.272$ $A_3 = 0.057$ $A_{-3} = 0.005$ $A_4 = 0.001$ $B_1 = 0.003$ $B_3 = 0.020$ $B_5 = 0.188$	$A_0 = 16.709$ $A_2 = -17.796$ $A_4 = 2.501$ $A_m = 0, m>4$ $B_1 = 0.020$ $B_3 = 0.096$ $B_5 = 0.884$ $B_m = 0, m>5$	

Table 4 - (continued)

E $\nu=6.000$ $\mu=16.784$	0 $\nu=5.500$ $\mu=14.012$	E $\nu=6.000$ $\mu=9.165$
$A_0 = 0.003$ $B_0 = 35.528$ $A_2 = 0.015$ $B_2 = -44.652$ $A_4 = 0.092$ $B_4 = 11.067$ $A_6 = 0.890$ $B_6 = -0.943$ $A_m = 0, m>6$ $B_m = 0, m>6$	$A_1 = 0.003$ $B_1 = 7.809$ $A_2 = 0.018$ $B_3 = -2.845$ $A_3 = 0.170$ $B_5 = 0.354$ $A_m = 0, m>3$ $B_7 = -0.006$	$A_1 = 0.118$ $B_1 = 6.600$ $A_3 = 0.324$ $B_3 = -6.158$ $A_5 = 0.972$ $B_5 = 0.972$ $A_m = 0, m>5$ $B_m = 0, m>5$
0 $\nu=5.507$ $\mu=7.060$	E $\nu=6.000$ $\mu=3.507$	0 $\nu=5.627$ $\mu=1.692$
$A_0 = -0.014$ $B_2 = 1.439$ $A_1 = -0.005$ $B_4 = -0.534$ $A_{-1} = -0.055$ $B_6 = 0.042$ $A_2 = -0.003$ $B_8 = 0.001$ $A_{-2} = -0.223$ $A_3 = 0.001$ $A_{-3} = 0.020$ $A_{-4} = 0.001$	$A_0 = 0.124$ $B_0 = 0.789$ $A_2 = 0.347$ $B_2 = 1.104$ $A_4 = 0.672$ $B_4 = -1.034$ $A_6 = -0.142$ $B_6 = 0.142$ $A_m = 0, m>6$ $B_m = 0, m>6$	$A_0 = -0.032$ $B_1 = 0.171$ $A_1 = 0.172$ $B_3 = 0.503$ $A_{-1} = -0.017$ $B_5 = -0.143$ $A_2 = 0.330$ $B_7 = 0.005$ $A_{-2} = 0.015$ $A_3 = -0.076$ $A_{-3} = -0.001$
E $\nu=6.000$ $\mu=0.000$	0 $\nu=5.927$ $\mu=-2\ 015$	E $\nu=6.000$ $\mu=-3.507$
$A_1 = 0.884$ $B_1 = 0.884$ $A_3 = 0.795$ $B_3 = 0.795$ $A_5 = -0.265$ $B_5 = -0.265$ $A_m = 0, m>5$ $B_m = 0, m>5$	$A_0 = -0.301$ $B_2 = 0.099$ $A_1 = -0.115$ $B_4 = 0.282$ $A_{-1} = -0.365$ $B_6 = -0.054$ $A_2 = 0.352$ $A_{-2} = 0.187$ $A_3 = -0.072$ $A_{-3} = -0.016$ $A_4 = 0.001$	$A_0 = 0.789$ $B_0 = 0.124$ $A_2 = 1.104$ $B_2 = 0.347$ $A_4 = -1.034$ $B_4 = 0.672$ $A_6 = 0.142$ $B_6 = -0.142$ $A_m = 0, m>6$ $B_m = 0, m>6$

Table 4 - (continued)

O ν=5.997 μ=-9.120		E ν=6.000 μ=-9.165		O ν=6.000 μ=-16.783	
A_0 = 0.552	B_1 = 0.008	A_1 = 6.600	B_1 = 0.118	A_0 =-8.470	B_2 = 0.002
A_1 = 1.376	B_3 = 0.045	A_3 =-6.158	B_3 = 0.324	A_1 = 6.670	B_4 = 0.016
A_{-1}=-1.217	B_5 = 0.197	A_5 = 0.972	B_5 = 0.972	A_{-1}= 4.225	B_6 = 0.155
A_2 =-0.878	B_7 =-0.017	A_m = 0,m>5	B_m = 0,m>5	A_2=-2.095	B_m = 0,m>6
A_{-2}= 0.367	B_9 =-0.001			A_{-2}=-0.820	
A_3 = 0.135				A_3 = 0.237	
A_{-3}=-0.026				A_{-3}= 0.047	
A_4 =-0.003				A_4 =-0.004	
				A_{-4}= 0.001	

E ν=6.000 μ=-16.784	
A_0 =35.528	B_0 = 0.003
A_2=-44.652	B_2 = 0.015
A_4 =11.067	B_4 = 0.092
A_6 =-0.943	B_6 = 0.890
A_m = 0,m>6	B_m = 0,m>6

E ν=7.000 μ=23.076		O ν=6.500 μ=19.804		E ν=7.000 μ=14.014	
A_1 = 0.003	B_0 =84.943	A_0 = 0.001	B_1 =14.855	A_0 = 0.022	B_1 =10.906
A_3 = 0.014	B_2=-110.135	A_1 = 0.001	B_3 =-6.317	A_2 = 0.089	B_3=-13.548
A_5 = 0.090	B_4 =29.321	A_2 = 0.002	B_5 = 1.102	A_4 = 0.310	B_5 = 4.005
A_7 = 0.894	B_6 =-2.715	A_3 = 0.012	B_7 =-0.058	A_6 = 0.993	B_7 =-0.363
A_m = 0,m>7	B_m = 0,m>6	A_4 = 0.124	B_9 =-0.001	A_m = 0,m>6	B_m = 0,m>7

Table 4 - (continued)

0 $\nu=6.501$ $\mu=11.440$	E $\nu=7.000$ $\mu=6.708$	0 $\nu=6.540$ $\mu=4.675$
$A_0 = 0.341$ $A_1 = 0.010$ $A_{-1} = 0.013$ $A_2 = 0.043$ $A_{-2} = 0.007$ $A_3 = 0.187$ $A_{-3} = -0.003$ $A_4 = -0.017$ $A_5 = -0.001$ $B_2 = 2.299$ $B_4 = -1.125$ $B_6 = 0.154$ $B_8 = -0.003$	$A_1 = 0.143$ $A_3 = 0.296$ $A_5 = 0.704$ $A_7 = -0.143$ $A_m = 0, m>7$ $B_0 = 1.693$ $B_2 = 1.763$ $B_4 = -2.446$ $B_6 = 0.403$ $B_m = 0, m>6$	$A_0 = 0.037$ $A_1 = 0.023$ $A_{-1} = -0.100$ $A_2 = 0.013$ $A_{-2} = -0.234$ $A_3 = -0.009$ $A_{-3} = 0.049$ $A_4 = 0.001$ $B_1 = 0.266$ $B_3 = 0.646$ $B_5 = -0.274$ $B_7 = 0.023$
E $\nu=7.000$ $\mu=1.770$	0 $\nu=6.792$ $\mu=-0.630$	E $\nu=7.000$ $\mu=-1.770$
$A_0 = 0.283$ $A_2 = 0.638$ $A_4 = 0.692$ $A_6 = -0.200$ $A_m = 0, m>6$ $B_1 = 0.852$ $B_3 = 0.611$ $B_5 = -0.534$ $B_7 = 0.071$ $B_m = 0, m>7$	$A_0 = -1.492$ $A_1 = 1.244$ $A_{-1} = -0.589$ $A_2 = 1.520$ $A_{-2} = 1.486$ $A_3 = -0.703$ $A_{-3} = -0.274$ $A_4 = 0.051$ $A_{-4} = 0.003$ $A_5 = 0.001$ $B_2 = 0.148$ $B_4 = 0.315$ $B_6 = -0.099$ $B_8 = 0.004$	$A_1 = 0.852$ $A_3 = 0.611$ $A_5 = -0.534$ $A_7 = 0.071$ $A_m = 0, m>7$ $B_0 = 0.283$ $B_2 = 0.638$ $B_4 = 0.692$ $B_6 = -0.200$ $B_m = 0, m>6$
0 $\nu=6.980$ $\mu=-6.500$	E $\nu=7.000$ $\mu=-6.708$	0 $\nu=7.000$ $\mu=-14.003$
$A_0 = -0.113$ $A_1 = -0.043$ $A_{-1} = -0.122$ $A_2 = 0.556$ $A_{-2} = 0.097$ $A_3 = -0.183$ $A_{-3} = -0.016$ $A_4 = 0.013$ $B_1 = -0.056$ $B_3 = 0.082$ $B_5 = 0.223$ $B_7 = -0.044$	$A_0 = 1.693$ $A_2 = 1.763$ $A_4 = -2.446$ $A_6 = 0.403$ $A_m = 0, m>6$ $B_1 = 0.143$ $B_3 = 0.296$ $B_5 = 0.704$ $B_7 = -0.143$ $B_m = 0, m>7$	$A_0 = 1.176$ $A_1 = 3.852$ $A_{-1} = -2.930$ $A_2 = -3.022$ $A_{-2} = 1.163$ $A_3 = 0.672$ $A_{-3} = -0.143$ $A_4 = -0.040$ $A_{-4} = 0.002$ $A_5 = -0.001$ $B_2 = 0.008$ $B_4 = 0.036$ $B_6 = 0.160$ $B_8 = -0.014$ $B_{10} = -0.001$

Table 4 - (continued)

E $\nu=7.000$ $\mu=-14.014$	O $\nu=7.000$ $\mu=-23.075$	E $\nu=7.000$ $\mu=-23.076$
$A_1=10.906$ $B_0=0.022$ $A_3=-13.548$ $B_2=0.089$ $A_5=4.005$ $B_4=0.310$ $A_7=-0.363$ $B_6=0.993$ $A_m=0, m>7$ $B_m=0, m>6$	$A_0=-17.244$ $B_3=0.004$ $A_1=13.906$ $B_5=0.023$ $A_{-1}=9.490$ $B_7=0.124$ $A_2=-5.147$ $A_{-2}=-2.333$ $A_3=0.816$ $A_{-3}=0.224$ $A_4=-0.040$ $A_{-4}=-0.003$ $A_5=-0.001$	$A_0=84.943$ $B_1=0.003$ $A_2=-110.135$ $B_3=0.014$ $A_4=29.321$ $B_5=0.090$ $A_6=-2.715$ $B_7=0.894$ $A_m=0, m>6$ $B_m=0, m>7$

E $\nu=8.000$ $\mu=30.368$	O $\nu=7.500$ $\mu=26.597$	E $\nu=8.000$ $\mu=19.880$
$A_2=0.002$ $B_0=89.332$ $A_4=0.013$ $B_2=-126.269$ $A_6=0.088$ $B_4=44.337$ $A_8=0.896$ $B_6=-7.370$ $A_m=0, m>8$ $B_8=0.970$ $B_m=0, m>8$	$A_0=0.001$ $B_1=28.305$ $A_1=0.001$ $B_3=-13.473$ $A_2=0.002$ $B_5=2.988$ $A_3=0.012$ $B_7=-0.266$ $A_4=0.124$ $B_9=0.004$ $A_m=0, m>4$	$A_1=0.025$ $B_1=24.157$ $A_3=0.080$ $B_3=-32.109$ $A_5=0.303$ $B_5=10.373$ $A_7=1.006$ $B_7=-1.008$ $A_m=0, m>7$ $B_m=0, m>7$
O $\nu=7.500$ $\mu=16.816$	E $\nu=8.000$ $\mu=11.036$	O $\nu=7.510$ $\mu=8.625$
$A_0=-0.025$ $B_2=3.889$ $A_1=-0.007$ $B_4=-2.329$ $A_{-1}=-0.007$ $B_6=0.458$ $A_2=-0.003$ $B_8=-0.025$ $A_{-2}=-0.032$ $A_3=-0.001$ $A_4=0.001$ $A_{-4}=0.013$ $A_{-5}=0.001$ $A_m=0, m>4$	$A_0=0.030$ $B_0=2.464$ $A_2=0.101$ $B_2=1.961$ $A_4=0.276$ $B_4=-4.877$ $A_6=0.739$ $B_6=1.597$ $A_8=-0.146$ $B_8=-0.145$ $A_m=0, m>8$ $B_m=0, m>8$	$A_0=-0.084$ $B_1=0.426$ $A_1=-0.031$ $B_3=0.910$ $A_{-1}=-0.047$ $B_5=-0.537$ $A_2=0.063$ $B_7=0.077$ $A_{-2}=-0.028$ $B_9=-0.001$ $A_3=0.164$ $A_{-3}=0.019$ $A_4=-0.033$ $A_{-4}=-0.001$

Table 4 - (continued)

E $\nu=8.000$ $\mu=4.334$	O $\nu=7.632$ $\mu=2.095$	E $\nu=8.000$ $\mu=0.000$
$A_1 = 0.369$ $B_1 = 1.622$ $A_3 = 0.530$ $B_3 = 0.726$ $A_5 = 0.696$ $B_5 = -1.115$ $A_7 = -0.181$ $B_7 = 0.181$ $A_m = 0, m>7$ $B_m = 0, m>7$	$A_0 = 2.224$ $B_2 = 0.213$ $A_1 = -0.209$ $B_4 = 0.363$ $A_{-1} = -0.188$ $B_6 = -0.166$ $A_2 = -0.098$ $B_8 = 0.015$ $A_{-2} = -0.277$ $A_3 = 0.162$ $A_{-3} = 0.109$ $A_4 = -0.026$ $A_{-4} = -0.007$	$A_0 = 0.273$ $B_0 = 0.273$ $A_2 = 0.562$ $B_2 = 0.562$ $A_4 = 0.469$ $B_4 = 0.469$ $A_6 = -0.348$ $B_6 = -0.348$ $A_8 = 0.044$ $B_8 = 0.044$ $A_m = 0, m>8$ $B_m = 0, m>8$
O $\nu=7.923$ $\mu=-3.685$	E $\nu=8.000$ $\mu=-4.334$	O $\nu=7.995$ $\mu=-10.905$
$A_0 = -0.035$ $B_1 = -0.330$ $A_1 = 0.003$ $B_3 = 0.157$ $A_{-1} = -0.010$ $B_5 = 0.303$ $A_2 = 0.002$ $B_7 = -0.100$ $A_{-2} = -0.370$ $B_9 = 0.005$ $A_3 = -0.002$ $A_{-3} = 0.120$ $A_{-4} = -0.008$	$A_1 = 1.622$ $B_1 = 0.369$ $A_3 = 0.726$ $B_3 = 0.530$ $A_5 = -1.115$ $B_5 = 0.696$ $A_7 = 0.181$ $B_7 = -0.181$ $A_m = 0, m>7$ $B_m = 0, m>7$	$A_0 = 0.067$ $B_2 = -0.040$ $A_1 = 0.031$ $B_4 = 0.066$ $A_{-1} = 0.072$ $B_6 = 0.184$ $A_2 = 0.654$ $B_8 = -0.036$ $A_{-2} = -0.080$ $A_3 = -0.300$ $A_{-3} = 0.019$ $A_4 = 0.039$ $A_{-4} = -0.001$ $A_5 = -0.001$
E $\nu=8.000$ $\mu=-11.036$	O $\nu=8.000$ $\mu=-19.877$	E $\nu=8.000$ $\mu=-19.880$
$A_0 = 2.464$ $B_0 = 0.030$ $A_2 = 1.961$ $B_2 = 0.101$ $A_4 = -4.877$ $B_4 = 0.276$ $A_6 = 1.597$ $B_6 = 0.739$ $A_8 = -0.145$ $B_8 = -0.146$ $A_m = 0, m>8$ $B_m = 0, m>8$	$A_0 = -1.674$ $B_1 = 0.021$ $A_1 = -8.728$ $B_3 = 0.006$ $A_{-1} = 4.318$ $B_5 = 0.028$ $A_2 = 7.926$ $B_7 = 0.133$ $A_{-2} = -2.097$ $B_9 = -0.012$ $A_3 = -2.297$ $B_{11} = -0.001$ $A_{-3} = 0.370$ $A_4 = 0.230$ $A_{-4} = -0.019$ $A_5 = -0.004$	$A_1 = 24.157$ $B_1 = 0.025$ $A_3 = -32.109$ $B_3 = 0.080$ $A_5 = 10.373$ $B_5 = 0.303$ $A_7 = -1.008$ $B_7 = 1.006$ $A_m = 0, m>7$ $B_m = 0, m>7$

Table 4 - (continued)

O $\nu=8.000$ $\mu=-30.368$		E $\nu=8.000$ $\mu=-30.368$	
$A_0=-38.665$ $A_1=31.492$ $A_{-1}=22.833$ $A_2=-13.184$ $A_{-2}=-6.699$ $A_3=2.663$ $A_{-3}=0.899$ $A_4=-0.221$ $A_{-4}=-0.039$ $A_5=0.003$	$B_4=0.002$ $B_6=0.011$ $B_8=0.116$	$A_0=89.332$ $A_2=-126.269$ $A_4=44.337$ $A_6=-7.370$ $A_8=0.970$ $A_m=0, m>8$	$B_2=0.002$ $B_4=0.013$ $B_6=0.088$ $B_8=0.896$ $B_m=0, m>8$

E $\nu=9.000$ $\mu=38.660$		O $\nu=8.500$ $\mu=34.389$		E $\nu=9.000$ $\mu=26.752$	
$A_3=0.002$ $A_5=0.013$ $A_7=0.087$ $A_9=0.898$ $A_m=0, m>9$	$B_0=833.672$ $B_2=-1197.523$ $B_4=438.450$ $B_6=-76.676$ $B_8=3.490$ $B_m=0, m>8$	$A_{-2}=-0.002$ $A_{-3}=-0.011$ $A_{-4}=-0.109$ $A_m=0, m<-4$	$B_1=48.438$ $B_3=-25.087$ $B_5=6.664$ $B_7=-0.831$ $B_9=0.034$	$A_0=0.004$ $A_2=0.019$ $A_4=0.075$ $A_6=0.299$ $A_8=1.016$ $A_m=0, m>8$	$B_1=50.366$ $B_3=-77.957$ $B_5=34.704$ $B_7=-6.362$ $B_9=0.249$ $B_m=0, m>9$
O $\nu=8.500$ $\mu=23.190$		E $\nu=9.000$ $\mu=16.438$		O $\nu=8.503$ $\mu=13.570$	
$A_0=0.087$ $A_1=0.022$ $A_{-1}=0.024$ $A_2=0.007$ $A_{-2}=0.009$ $A_3=0.030$ $A_{-3}=0.005$ $A_4=0.142$ $A_{-4}=-0.002$ $A_5=-0.013$ $A_6=-0.001$ $A_m=0, m<-4$	$B_2=6.709$ $B_4=-4.674$ $B_6=1.195$ $B_8=-0.113$ $B_{10}=0.002$	$A_1=0.033$ $A_3=0.084$ $A_5=0.268$ $A_7=0.765$ $A_9=-0.149$ $A_m=0, m>9$	$B_0=5.802$ $B_2=3.126$ $B_4=-11.245$ $B_6=4.147$ $B_8=-0.417$ $B_m=0, m>8$	$A_0=0.087$ $A_1=0.030$ $A_{-1}=0.032$ $A_2=0.016$ $A_{-2}=-0.056$ $A_3=0.010$ $A_{-3}=-0.157$ $A_4=-0.031$ $A_{-4}=0.031$	$B_1=0.727$ $B_3=1.474$ $B_5=-1.142$ $B_7=0.241$ $B_9=-0.014$

Table 4 - (continued)

E $\nu=9.000$ $\mu=8.004$	0 $\nu=8.545$ $\mu=5.630$	E $\nu=9.000$ $\mu=2.158$
$A_0 = 0.098$ $B_1 = 1.936$ $A_2 = 0.265$ $B_3 = 0.460$ $A_4 = 0.490$ $B_5 = -2.027$ $A_6 = 0.741$ $B_7 = 0.689$ $A_8 = -0.180$ $B_9 = -0.059$ $A_m = 0, m>8$ $B_m = 0, m>9$	$A_0 = -0.146$ $B_2 = 0.314$ $A_1 = -0.066$ $B_4 = 0.446$ $A_{-1} = -0.522$ $B_6 = -0.292$ $A_2 = 0.045$ $B_8 = 0.043$ $A_{-2} = -0.270$ $B_{10} = -0.001$ $A_3 = 0.078$ $A_{-3} = 0.358$ $A_4 = -0.028$ $A_{-4} = -0.052$ $A_5 = 0.001$	$A_1 = 0.365$ $B_0 = 0.521$ $A_3 = 0.440$ $B_2 = 0.942$ $A_5 = 0.442$ $B_4 = 0.464$ $A_7 = -0.280$ $B_6 = -0.607$ $A_9 = 0.033$ $B_8 = 0.093$ $A_m = 0, m>9$ $B_m = 0, m>8$
0 $\nu=8.787$ $\mu=-0.770$	E $\nu=9.000$ $\mu=-2.158$	0 $\nu=8.980$ $\mu=-7.710$
$A_0 = -0.211$ $B_1 = 10.313$ $A_1 = -0.075$ $B_3 = -1.546$ $A_{-1} = 0.046$ $B_5 = -2.326$ $A_2 = -0.024$ $B_7 = 1.138$ $A_{-2} = 0.045$ $B_9 = -0.111$ $A_3 = 0.356$ $B_{11} = -0.001$ $A_{-3} = -0.033$ $A_4 = 0.005$ $A_{-4} = 0.006$	$A_0 = 0.521$ $B_1 = 0.365$ $A_2 = 0.942$ $B_3 = 0.440$ $A_4 = 0.464$ $B_5 = 0.442$ $A_6 = -0.607$ $B_7 = -0.280$ $A_8 = 0.093$ $B_9 = 0.033$ $A_m = 0, m>8$ $B_m = 0, m>9$	$A_0 = -0.291$ $B_2 = -0.269$ $A_1 = -0.060$ $B_4 = 0.143$ $A_{-1} = -0.090$ $B_6 = 0.275$ $A_2 = -0.047$ $B_8 = -0.092$ $A_{-2} = -0.346$ $B_{10} = 0.005$ $A_3 = 0.064$ $A_{-3} = 0.169$ $A_4 = -0.016$ $A_{-4} = -0.022$ $A_5 = 0.001$
E $\nu=9.000$ $\mu=-8.004$	0 $\nu=9.000$ $\mu=-16.420$	E $\nu=9.000$ $\mu=-16.438$
$A_1 = 1.936$ $B_0 = 0.098$ $A_3 = 0.460$ $B_2 = 0.265$ $A_5 = -2.027$ $B_4 = 0.490$ $A_7 = 0.689$ $B_6 = 0.741$ $A_9 = -0.059$ $B_8 = -0.180$ $A_m = 0, m>9$ $B_m = 0, m>8$	$A_0 = 0.136$ $B_1 = -0.099$ $A_1 = 0.080$ $B_3 = -0.036$ $A_{-1} = 0.155$ $B_5 = 0.060$ $A_2 = 0.718$ $B_7 = 0.173$ $A_{-2} = -0.225$ $B_9 = -0.034$ $A_3 = -0.426$ $A_{-3} = 0.073$ $A_4 = 0.080$ $A_{-4} = -0.078$ $A_5 = -0.004$	$A_0 = 5.802$ $B_1 = 0.033$ $A_2 = 3.126$ $B_3 = 0.084$ $A_4 = -11.245$ $B_5 = 0.268$ $A_6 = 4.147$ $B_7 = 0.765$ $A_8 = -0.417$ $B_9 = -0.149$ $A_m = 0, m>8$ $B_m = 0, m>9$

Table 4 - (continued)

O $\nu=9.000$ $\mu=-26.751$	
$A_0=-0.218$	$B_2=0.017$
$A_1=-2.958$	$B_4=0.005$
$A_{-1}=0.548$	$B_6=0.024$
$A_2=2.979$	$B_8=0.114$
$A_{-2}=-0.311$	$B_{10}=-0.010$
$A_3=-1.056$	
$A_{-3}=0.071$	
$A_4=0.154$	
$A_{-4}=-0.006$	
$A_5=-0.007$	

E $\nu=9.000$ $\mu=-26.752$	
$A_1=50.366$	$B_0=0.004$
$A_3=-77.957$	$B_2=0.019$
$A_5=34.704$	$B_4=0.075$
$A_7=-6.362$	$B_6=0.299$
$A_9=0.249$	$B_8=1.016$
$A_m=0, m>9$	$B_m=0, m>8$

O $\nu=9.000$ $\mu=-38.660$	
$A_0=-89.784$	$B_5=0.001$
$A_1=73.229$	$B_7=0.010$
$A_{-1}=55.812$	$B_9=0.103$
$A_2=-33.698$	$B_m=0, m>9$
$A_{-2}=-18.771$	
$A_3=8.177$	
$A_{-3}=3.216$	
$A_4=-0.951$	
$A_{-4}=-0.236$	
$A_5=0.037$	
$A_{-5}=0.003$	

E $\nu=9.000$ $\mu=-38.660$	
$A_0=833.672$	$B_3=0.002$
$A_2=-1197.523$	$B_5=0.013$
$A_4=438.450$	$B_7=0.087$
$A_6=-76.676$	$B_9=0.898$
$A_8=3.490$	$B_m=0, m>9$
$A_m=0, m>8$	

Table 5 - Lowest Eigenvalues of the Even Dirichlet and Odd Neumann Problem for $k^2 = 0.1$ and $k^2 = 0.9$

Even Dirichlet			
$k^2 = 0.1$	$k'^2 = 0.9$	$k^2 = 0.9$	$k'^2 = 0.1$
ν	μ	ν	μ
0.407	0.208	0.171	0.010

Odd Neumann			
$k^2 = 0.1$	$k'^2 = 0.9$	$k^2 = 0.9$	$k'^2 = 0.1$
ν	μ	ν	μ
0.613	0.188	0.970	-0.806

CHAPTER IV

SOLUTION OF THE VECTOR WAVE EQUATION

In this chapter the electric field in a source-free region containing a perfectly-conducting plane angular sector is determined. The medium surrounding the plane angular sector is linear, isotropic, and homogeneous. Thus $\overline{E}$ satisfies the equation

$$\nabla \times \nabla \times \overline{E} - \kappa^2 \overline{E} = 0 \tag{4.1}$$

with the boundary condition

$$\hat{n} \times \overline{E} = 0 \tag{4.2}$$

on the surface of the plane angular sector and a radiation condition as $r \to \infty$. The unit vector $\hat{n}$ is normal to the plane angular sector. Since $\hat{n}$ is not defined at the edges and tip of the plane angular sector, there are also conditions which must be satisfied there. These will be discussed in Chapter VI.

Following the method described by Morse and Feshbach[15] define the following three independent vector wave functions.

$$\overline{L} = \nabla\psi \tag{4.3}$$

$$\overline{M}_2 = \nabla \times \psi_2 \overline{R} \tag{4.4}$$

$$\overline{N}_1 = \frac{1}{\kappa} \nabla \times \nabla \times \psi_1 \overline{R} \tag{4.5}$$

where ψ, ψ_1, and ψ_2 are scalar wave functions and $\bar{R} = r\,\hat{r}$ is a radius vector. Since $\nabla \cdot \bar{E} = 0$, it is clear that only the latter two can be solutions of equation (4.1). Checking $\bar{M}_2$ and $\bar{N}_1$ in equation (4.2), it is seen that ψ_2 must satisfy the Neumann boundary condition and ψ_1 the Dirichlet boundary condition. Thus ψ_1 and ψ_2 are the complete sets of scalar wave functions determined in Chapter III. Since the sets ψ_1 and ψ_2 are complete, the sets $\bar{M}_2$ and $\bar{N}_1$ are complete, and thus any divergenceless vector can be expanded as a sum of these functions.

Performing the indicated curl operations in equations (4.4) and (4.5), the vector wave functions can be written in component forms as

$$\bar{M}_{\substack{e\\o}\ell 2} = \frac{z_{\nu_{\ell 2}}(\kappa r)}{\sqrt{k^2 \sin^2\theta + k'^2 \sin^2\phi}} \left[\sqrt{1 - k^2\cos^2\theta}\; \Theta'_{\substack{e\\o}\ell 2}(\theta)\, \Phi_{\substack{e\\o}\ell 2}(\phi)\,\hat{\phi} - \sqrt{1 - k'^2\cos^2\phi}\; \Theta_{\substack{e\\o}\ell 2}(\theta)\, \Phi'_{\substack{e\\o}\ell 2}(\phi)\,\hat{\theta} \right] \tag{4.6}$$

$$\begin{aligned}\bar{N}_{\substack{e\\o}\ell 1} &= \nu_{\ell 1}(\nu_{\ell 1}+1)\,\frac{z_{\nu_{\ell 1}}(\kappa r)}{\kappa r}\, \Theta_{\substack{e\\o}\ell 1}(\theta)\, \Phi_{\substack{e\\o}\ell 1}(\phi)\,\hat{r} \\ &+ \frac{(r\, z_{\nu_{\ell 1}}(\kappa r))'}{\kappa r\sqrt{k^2\sin^2\theta + k'^2\sin^2\phi}} \left[\sqrt{1 - k'^2\cos^2\phi}\; \Theta_{\substack{e\\o}\ell 1}(\theta)\, \Phi'_{\substack{e\\o}\ell 1}(\phi)\,\hat{\phi} + \sqrt{1 - k^2\cos^2\theta}\; \Theta'_{\substack{e\\o}\ell 1}(\theta)\, \Phi_{\substack{e\\o}\ell 1}(\phi)\,\hat{\theta} \right]\end{aligned} \tag{4.7}$$

The subscript ℓ is an ordering index. It is an integer which identifies each (ν_ℓ, μ_ℓ) eigenvalue pair, and is useful not only for identification but as a summation index.

$z_{\nu_\ell}(\kappa r)$ is a spherical Bessel function, its type depending on the boundary condition on r. All primes indicate differentation with respect to the independent variables r, θ, and ϕ.

Note that $(\nu=0, \mu=0)$ is an eigenvalue pair for the scalar wave functions, but it is not for the vector wave functions. Since $\overline{M}_{e\ell 2}$ contains derivatives of the scalar functions and the scalar eigenfunctions that correspond to $(\nu=0, \mu=0)$ are constant, $\overline{M}_{e2}(\nu=0, \mu=0)$ vanishes.

It is also possible to define two other vector wave functions, $\overline{M}_{\substack{e\\o}\ell 1}$, and $\overline{N}_{\substack{e\\o}\ell 2}$. These do not satisfy equation (4.2) so they are not acceptable as solutions of this problem. They are useful, however, and will be used in the dyadic Green's function derived in the next chapter. These functions are defined as follows:

$$\overline{M}_{\substack{e\\o}\ell 1} = \frac{z_{\nu_{\ell 1}}(\kappa r)}{\sqrt{k^2 \sin^2\theta + k'^2 \sin^2\phi}} \left[\sqrt{1 - k^2\cos^2\theta}\ \Theta'_{\substack{e\\o}\ell 1}(\theta)\ \Phi_{\substack{e\\o}\ell 1}(\phi)\ \hat{\phi} - \sqrt{1 - k'^2\cos^2\phi}\ \Theta_{\substack{e\\o}\ell 1}(\theta)\ \Phi'_{\substack{e\\o}\ell 1}(\phi)\ \hat{\theta} \right] \tag{4.8}$$

$$\begin{aligned}\overline{N}_{\substack{e\\o}\ell 2} = {} & \nu_{\ell 2}(\nu_{\ell 2}+1)\,\frac{z_{\nu_{\ell 2}}(\kappa r)}{\kappa r}\ \Theta_{\substack{e\\o}\ell 2}(\theta)\ \Phi_{\substack{e\\o}\ell 2}(\phi)\ \hat{r} \\ & + \frac{(r\, z_{\nu_{\ell 2}}(\kappa r))'}{\kappa r\sqrt{k^2\sin^2\theta + k'^2\sin^2\phi}} \left[\sqrt{1 - k'^2\cos^2\phi}\ \Theta_{\substack{e\\o}\ell 2}(\theta)\ \Phi'_{\substack{e\\o}\ell 2}(\phi)\ \hat{\phi} + \sqrt{1 - k^2\cos^2\theta}\ \Theta'_{\substack{e\\o}\ell 2}(\theta)\ \Phi_{\substack{e\\o}\ell 2}(\phi)\ \hat{\theta} \right].\end{aligned} \tag{4.9}$$

These are related to $\overline{M}_{\substack{e\\o}\ell 2}$ and $\overline{N}_{\substack{e\\o}\ell 1}$ in the following manner:

$$\overline{M}_{\substack{e\\o}\ell 1} = \frac{1}{\kappa} \nabla \times \overline{N}_{\substack{e\\o}\ell 1} \tag{4.10}$$

$$\overline{N}_{\substack{e\\o}\ell 2} = \frac{1}{\kappa} \nabla \times \overline{M}_{\substack{e\\o}\ell 2} \tag{4.11}$$

$$\overline{M}_{\substack{e\\o}\ell 2} = \frac{1}{\kappa} \nabla \times \overline{N}_{\substack{e\\o}\ell 2} \tag{4.12}$$

$$\overline{N}_{\substack{e\\o}\ell 1} = \frac{1}{\kappa} \nabla \times \overline{M}_{\substack{e\\o}\ell 1} \tag{4.13}$$

All four of the vector wave functions defined by equations (4.6), (4.7), (4.8), and (4.9) possess orthogonality properties on the surface of a sphere. These will be discussed in the next chapter, where they are used to find the dyadic Green's function. To facilitate the investigation of orthogonality, the following auxiliary vector wave functions are introduced at this point.

$$\begin{aligned} \overline{m}_{\substack{e\\o}\ell 1} = {} & \frac{\sqrt{1 - k^2 \cos^2 \theta}}{\sqrt{k^2 \sin^2 \theta + k'^2 \sin^2 \phi}} \; \Theta'_{\substack{e\\o}\ell 1}(\theta) \, \Phi_{\substack{e\\o}\ell 1}(\phi) \, \hat{\phi} \\ & - \frac{\sqrt{1 - k'^2 \cos^2 \phi}}{\sqrt{k^2 \sin^2 \theta + k'^2 \sin^2 \phi}} \; \Theta_{\substack{e\\o}\ell 1}(\theta) \, \Phi'_{\substack{e\\o}\ell 1}(\phi) \, \hat{\theta} \end{aligned} \tag{4.14}$$

$$\begin{aligned} \hat{r} \times \overline{m}_{\substack{e\\o}\ell 1} = {} & \frac{\sqrt{1 - k'^2 \cos^2 \phi}}{\sqrt{k^2 \sin^2 \theta + k'^2 \sin^2 \phi}} \; \Theta_{\substack{e\\o}\ell 1}(\theta) \, \Phi'_{\substack{e\\o}\ell 1}(\phi) \, \hat{\phi} \\ & + \frac{\sqrt{1 - k^2 \cos^2 \theta}}{\sqrt{k^2 \sin^2 \theta + k'^2 \sin^2 \phi}} \; \Theta'_{\substack{e\\o}\ell 1}(\theta) \, \Phi_{\substack{e\\o}\ell 1}(\phi) \, \hat{\theta} \end{aligned} \tag{4.15}$$

$$\bar{\ell}_{\substack{e\\o}\ell 1} = \nu_{\ell 1}(\nu_{\ell 1}+1)\ \Theta_{\substack{e\\o}\ell 1}(\theta)\ \Phi_{\substack{e\\o}\ell 1}(\phi)\ \hat{r} \tag{4.16}$$

The auxiliary vector wave functions $\bar{m}_{\substack{e\\o}\ell 2}$, $\hat{r} \times \bar{m}_{\substack{e\\o}\ell 2}$, $\bar{\ell}_{\substack{e\\o}\ell 2}$ are defined similarly except that Neumann functions are used instead of the Dirichlet. These functions are related to the original vector wave functions in the following manner.

$$\bar{M}_{\substack{e\\o}\ell 2} = z_{\nu_{\ell 2}}(\kappa r)\ \bar{m}_{\substack{e\\o}\ell 2} \tag{4.17}$$

$$\bar{N}_{\substack{e\\o}\ell 1} = \frac{z_{\nu_{\ell 1}}(\kappa r)}{\kappa r}\ \bar{\ell}_{\substack{e\\o}\ell 1} + \frac{(r\ z_{\nu_{\ell 1}}(\kappa r))'}{\kappa r}\ (\hat{r} \times \bar{m}_{\substack{e\\o}\ell 1}) \tag{4.18}$$

and similarly for $\bar{M}_{\substack{e\\o}\ell 1}$ and $\bar{N}_{\substack{e\\o}\ell 2}$.

For a more complete discussion of vector wave functions, see Morse and Feshbach[15]. Vector wave functions are also discussed by Stratton[16], Van Bladel[17], and Spence and Wells[18].

CHAPTER V

DETERMINATION OF THE DYADIC GREEN'S FUNCTION

In this chapter, the dyadic Green's function is derived. The derivation is given in some detail since the Green's function is composed of previously unknown vector wave functions. In the process, some properties of these vector wave functions are determined.

Introduction

In order to find the total field due to an arbitrary current $\overline{J}$ in the vicinity of the perfectly conducting plane angular sector, the following set of equations must be solved.

$$\nabla \times \nabla \times \overline{E} - \kappa^2 \overline{E} = - j\omega\mu\overline{J} \tag{5.1}$$

$$\hat{n} \times \overline{E} = 0 \tag{5.2}$$

on the plane angular sector, $\overline{E}$ must satisfy the radiation condition as $r \to \infty$, and as was mentioned in Chapter IV, there are also edge and tip conditions which must be satisfied.

The solution to this set of equations can be written

$$\overline{E}(\overline{R}) = j\omega\mu \int_V \overline{\overline{\Gamma}}(\overline{R},\overline{R}') \cdot \overline{J}(\overline{R}')\, dv + \int_S [\overline{E}(\overline{R}') \times \hat{n} \cdot \nabla' \times \overline{\overline{\Gamma}}(\overline{R}',\overline{R})]\, ds \tag{5.3}$$

where $\overline{R}$ is the field point, $\overline{R}'$ is the source point, and v is a volume enclosing the source current $\overline{J}$. The surface S is the plane angular sector. Ordinarily the surface integral is zero because of equation (5.2). However, if the plane angular sector contains slots, $\hat{n} \times \overline{E}$ will not vanish in the aperture of the slots. Thus the dyadic Green's function obtained in this paper can be used not only to determine the fields radiated by an arbitrary current distribution in the vicinity of a plane angular sector, but also to determine the fields radiated by slots in a plane angular sector. $\overline{\overline{\Gamma}}(\overline{R},\overline{R}')$ is the dyadic Green's function. It must satisfy the following equations.

$$\nabla \times \nabla \times \overline{\overline{\Gamma}}(\overline{R},\overline{R}') - \kappa^2 \overline{\overline{\Gamma}}(\overline{R},\overline{R}') = - \overline{\overline{\varepsilon}}\, \delta(|\overline{R}-\overline{R}'|) \tag{5.4}$$

where $\overline{\overline{\varepsilon}}$ is the unit dyadic.

$$\hat{n} \times \overline{\overline{\Gamma}}(\overline{R},\overline{R}') = 0 \tag{5.5}$$

on the plane angular sector, $\overline{\overline{\Gamma}}(\overline{R},\overline{R}')$ must satisfy a radiation condition as $r \to \infty$, and $\overline{\overline{\Gamma}}(\overline{R},\overline{R}')$ must satisfy conditions at the edges and tip.

In order to make the derivation of $\overline{\overline{\Gamma}}(\overline{R},\overline{R}')$ as simple as possible, consider the following operation on equations (5.4) and (5.5). Let an arbitrary unit vector, $\hat{a}$, be dot multiplied to the right of the two equations. The result is

$$\nabla \times \nabla \times \overline{G}(\overline{R},\overline{R}') - \kappa^2 \overline{G}(\overline{R},\overline{R}') = - \hat{a}\, \delta(|\overline{R}-\overline{R}'|) \tag{5.6}$$

$$\hat{n} \times \overline{G}(\overline{R},\overline{R}') = 0 \tag{5.7}$$

where

$$\overline{G}(\overline{R},\overline{R}') = \overline{\overline{\Gamma}}(\overline{R},\overline{R}') \cdot \hat{a} \tag{5.8}$$

The vector Green's function $\overline{G}(\overline{R},\overline{R}')$ is the field due to a unit vector point source $\hat{a}$ located at $\overline{R}'$. Once $\overline{G}(\overline{R},\overline{R}')$ is determined it can be compared with equation (5.8) to determine $\overline{\overline{\Gamma}}(\overline{R},\overline{R}')$.

It was determined in the last chapter that an arbitrary divergenceless vector function could be represented as a sum over the complete set of vector wave functions, $\overline{M}_{\substack{e\\o}\ell 2}$, and $\overline{N}_{\substack{e\\o}\ell 1}$. Thus $\overline{G}(\overline{R},\overline{R}')$ can be written

$$\overline{G}(\overline{R},\overline{R}') = \sum_q [a_q(\overline{R}')\,\overline{M}_{q2}(\overline{R}) + b_q(\overline{R}')\,\overline{N}_{q1}(\overline{R})] \tag{5.9}$$

where the subscript q replaces the $\substack{e\\o}\ell$ subscript used previously, and is meant to include all of these functions. The subscript 1 or 2 means, as before, that Dirichlet or Neumann functions, respectively, are used in the vector wave functions. Equation (5.9) automatically satisfies equation (5.7). The coefficients $a_q(\overline{R}')$ and $b_q(\overline{R}')$ will be determined in order to satisfy equation (5.6).

Derivation

The singularity at $\overline{R}'$ separates space into two regions. Let the dividing surface S' be a sphere with radius r' except for an infinitesimal cut around the quarter plane. (See Figure 7) The volume within S' is called region I and that outside of S' is called region II. Also define a surface S with the same shape as S' but at a radius r. The surface S is in region II. Within region I, the vector Green's function has the boundary condition on r that the function must be regular at r = 0. This implies that the Bessel functions used in the vector wave functions must be of the first kind. Thus in region I,

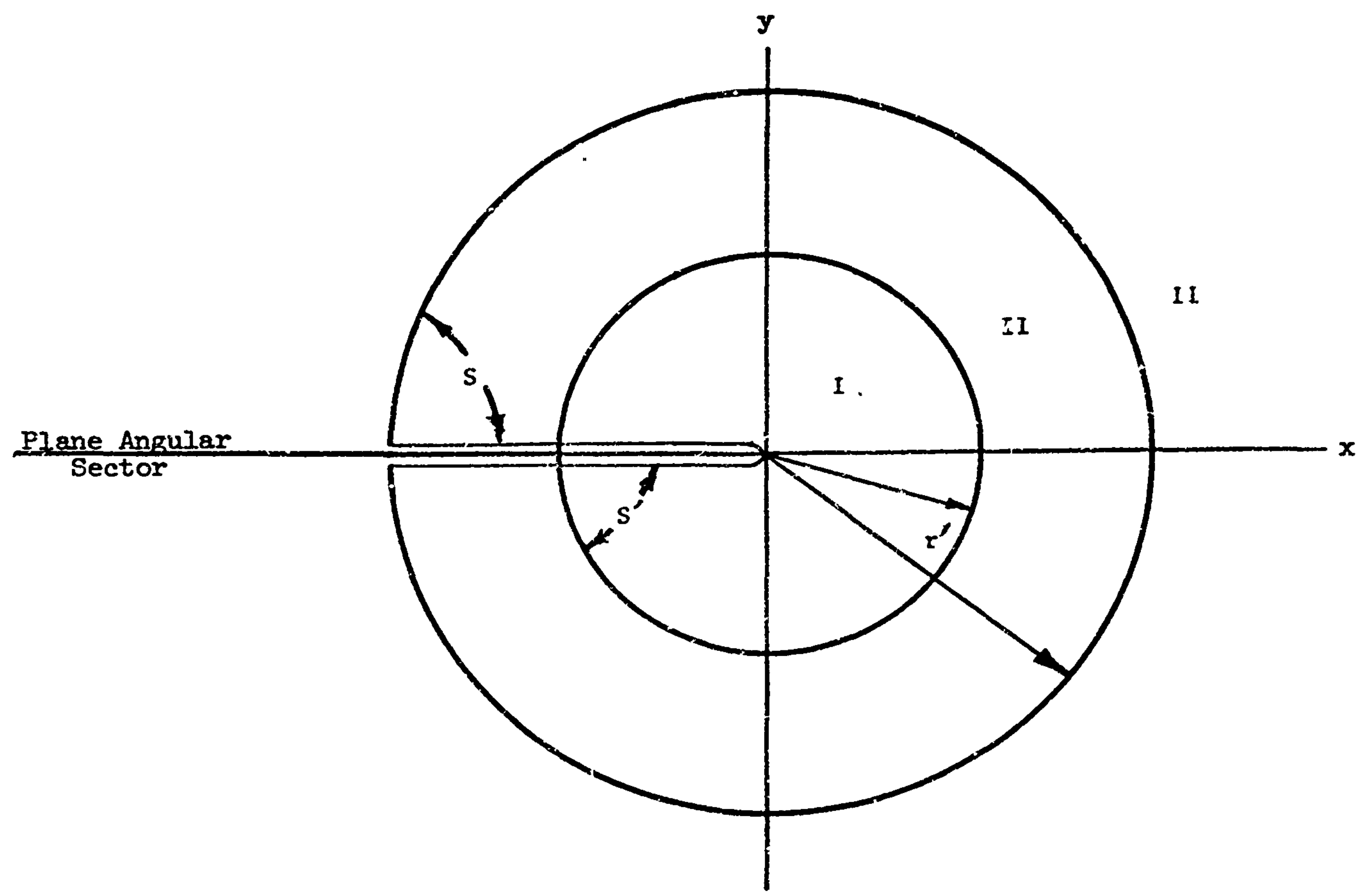

Fig. 7. Regions used in the determination of $\overline{G}(\overline{R},\overline{R}')$

$$\overline{G}^{I} (\overline{R},\overline{R}') = \sum_q [a_q (\overline{R}') \overline{M}^{I}_{q2} (\overline{R}) + b_q (\overline{R}') \overline{N}^{I}_{q1} (\overline{R})] \tag{5.10}$$

where the superscript I means that $z_\nu (\kappa r) = j_\nu (\kappa r)$. In region II, the boundary condition on r is that the function must represent an outward propagating wave for large r. This implies that the Bessel functions must be Hankel functions of the second kind. Thus in region II

$$\overline{G}^{II} (\overline{R},\overline{R}') = \sum_q [a_q (\overline{R}') \overline{M}^{II}_{q2} (\overline{R}) + b_q (\overline{R}') \overline{N}^{II}_{q1} (\overline{R})] \tag{5.11}$$

where the superscript II means that $z_\nu (\kappa r) = h_\nu^{(2)} (\kappa r)$.

In order to determine the unknown coefficients $a_q (\overline{R}')$ and $b_q (\overline{R}')$, consider the following equation, known as Green's second identity for vectors.

$$\int_V (\overline{B} \cdot \nabla \times \nabla \times \overline{A} - \overline{A} \cdot \nabla \times \nabla \times \overline{B})\, dv \tag{5.12}$$

$$= \int_S (\overline{A} \times \nabla \times \overline{B} - \overline{B} \times \nabla \times \overline{A}) \cdot \hat{n}\, ds$$

where $\hat{n}$ is a unit normal to the surface S which is pointing out of the enclosed volume V. Let the surface be the same as the surface S in Figure 7. Then V includes all of region I and part of region II. Let

$$\overline{A} = \overline{G} (\overline{R},\overline{R}') \tag{5.13}$$

$$\overline{B} = \overline{M}^{I}_{p2} (\overline{R}) \tag{5.14}$$

The volume integral is readily evaluated. From equations (5.6) and (5.13),

$$\nabla \times \nabla \times \overline{A} = \nabla \times \nabla \times \overline{G}\ (\overline{R},\overline{R}') = \kappa^2\ \overline{G}\ (\overline{R},\overline{R}') - \hat{a}\ \delta\ (|\overline{R}-\overline{R}'|) \tag{5.15}$$

From equations (4.1) and (5.14),

$$\nabla \times \nabla \times \overline{B} = \nabla \times \nabla \times \overline{M}^{I}_{p2}\ (\overline{R}) = \kappa^2\ \overline{M}^{I}_{p2}\ (\overline{R}) \tag{5.16}$$

Thus the left hand side of equation (5.12) is

$$\int_v [\overline{M}^{I}_{p2}\ (\overline{R}) \cdot \kappa^2\ \overline{G}\ (\overline{R},\overline{R}') - \overline{M}^{I}_{p2}\ (\overline{R}) \cdot \hat{a}\ \delta\ (|\overline{R}-\overline{R}'|) \tag{5.17}$$

$$- \overline{G}\ (\overline{R},\overline{R}') \cdot \kappa^2\ \overline{M}^{I}_{p2}\ (\overline{R})]\ dv = - \hat{a} \cdot \overline{M}^{I}_{p2}\ (\overline{R}')$$

The evaluation of the surface integral is not quite as simple. Consider the following vector identities.

$$\overline{A} \times \nabla \times \overline{B} \cdot \hat{n} = \hat{n} \cdot \overline{A} \times \nabla \times \overline{B} = \hat{n} \times \overline{A} \cdot \nabla \times \overline{B} \tag{5.18a}$$

$$\overline{B} \times \nabla \times \overline{A} \cdot \hat{n} = \hat{n} \cdot \overline{B} \times \nabla \times \overline{A} = \hat{n} \times \overline{B} \cdot \nabla \times \overline{A} \tag{5.18b}$$

Using these in the right hand side of equation (5.12), the following is obtained.

$$\int_s [\hat{n} \times \overline{G}\ (\overline{R},\overline{R}') \cdot \nabla \times \overline{M}^{I}_{p2}\ (R) \tag{5.19}$$

$$- \hat{n} \times \overline{M}^{I}_{p2}\ (\overline{R}) \cdot \nabla \times \overline{G}\ (\overline{R},\overline{R}')]\ ds$$

On the part of S that encloses the quarter plane

$$\hat{n} \times \overline{G}\ (\overline{R},\overline{R}') = 0 \tag{5.20}$$

and

$$\hat{n} \times \overline{M}^{I}_{p2}\ (\overline{R}) = 0 \tag{5.21}$$

so that the integration is over the sphere with radius r. On this sphere

$$\hat{n} \times \bar{G}(\bar{R},\bar{R}') = \hat{r} \times \bar{G}^{II}(\bar{R},\bar{R}') = \sum_q [a_q(\bar{R}')(\hat{r} \times \bar{M}^{II}_{q2}(\bar{R})) \tag{5.22}$$

$$+ b_q(\bar{R}')(\hat{r} \times \bar{N}^{II}_{q1}(\bar{R}))]$$

$$\hat{n} \times \bar{M}^{I}_{p2}(\bar{R}) = \hat{r} \times \bar{M}^{I}_{p2}(\bar{R}) \tag{5.23}$$

and with the help of equations (4.10) and (4.11)

$$\nabla \times \bar{M}^{I}_{p2}(\bar{R}) = \kappa\, \bar{N}^{I}_{p2}(\bar{R}) \tag{5.24}$$

$$\nabla \times \bar{G}(\bar{R},\bar{R}') = \nabla \times \bar{G}^{II}(\bar{R},\bar{R}') \tag{5.25}$$

$$= \kappa \sum_q [a_q(\bar{R}')\, \bar{N}^{II}_{q2}(\bar{R}) + b_q(\bar{R}')\, \bar{M}^{II}_{q1}(\bar{R})]$$

Using these results, the surface integral is

$$\kappa \int_s \sum_q \{a_q(\bar{R}')\, [\hat{r} \times \bar{M}^{II}_{q2}(\bar{R}) \cdot \bar{N}^{I}_{p2}(\bar{R}) - \hat{r} \times \bar{M}^{I}_{p2}(\bar{R}) \cdot \bar{N}^{II}_{q2}(\bar{R})] \tag{5.26}$$

$$+ b_q(\bar{R}')\, [\hat{r} \times \bar{N}^{II}_{q1}(\bar{R}) \cdot \bar{N}^{I}_{p2}(\bar{R}) - \hat{r} \times \bar{M}^{I}_{p2}(\bar{R}) \cdot \bar{M}^{II}_{q1}(\bar{R})]\}\, ds$$

$$= \kappa \sum_q [a_q(\bar{R}')\, A_q + b_q(\bar{R}')\, B_q]$$

First consider B_q

$$B_q = \int_s [\hat{r} \times \bar{N}^{II}_{q1}(\bar{R}) \cdot \bar{N}^{I}_{p2}(\bar{R}) - \hat{r} \times \bar{M}^{I}_{p2}(\bar{R}) \cdot \bar{M}^{II}_{q1}(\bar{R})]\, ds \tag{5.27}$$

Using the auxiliary vector wave functions introduced in the previous chapter, B_q can be written

$$B_q = -\left[\frac{(r\, h^{(2)}_{\nu_{q1}}(\kappa r))'\,(r\, j_{\nu_{p2}}(\kappa r))' \quad + r^2\, j_{\nu_{p2}}(\kappa r)\, h_{\nu_{q1}}(\kappa r)}{\kappa^2}\right] \tag{5.28}$$

$$\int_0^{\pi}\int_{-\pi}^{\pi} (\overline{m}_{q1} \cdot \hat{r} \times \overline{m}_{p2})\, \rho\, d\,\phi d\,\theta = f\,(r)\, C_q$$

where the weight function ρ is

$$\rho = \frac{h_\theta h_\phi}{r^2} = \frac{k^2 \sin^2\theta + k'^2 \sin^2\phi}{\sqrt{1 - k^2\cos^2\theta}\,\sqrt{1 - k'^2\cos^2\phi}} \tag{5.29}$$

Using the definitions of the auxiliary wave functions, C_q is

$$C_q = \int_0^{\pi}\int_{-\pi}^{\pi} [\Theta'_{q1}(\theta)\, \Phi_{q1}(\phi)\, \Theta_{p2}(\theta)\, \Phi'_{p2}(\phi) \tag{5.30}$$

$$- \Theta_{q1}(\theta)\, \Phi'_{q1}(\phi)\, \Theta'_{p2}(\theta)\, \Phi_{p2}(\phi)]\, d\phi\, d\theta$$

If the q1 functions and p2 functions are both even or both odd, the integrand is odd with respect to ϕ and $C_q = 0$. Thus the q1 functions must be even and the p2 functions odd or vice versa if C_q is to be nonzero. Separating the θ and ϕ integrals C_q can be written

$$C_q = \int_{-\pi}^{\pi} \Phi_{q1}(\phi)\, \Phi'_{p2}(\phi)\, d\phi \int_0^{\pi} \Theta'_{q1}(\theta)\, \Theta_{p2}(\theta)\, d\theta \tag{5.31}$$

$$- \int_{-\pi}^{\pi} \Phi'_{q1}(\phi)\, \Phi_{p2}(\phi)\, d\phi \int_0^{\pi} \Theta_{q1}(\theta)\, \Theta'_{p2}(\theta)\, d\theta$$

Integrating the first θ integral by parts, the following is obtained.

$$\int_0^\pi \Theta'_{q1}(\theta)\ \Theta_{p2}(\theta)\ d\theta = \Theta_{q1}(\theta)\ \Theta_{p2}(\theta)\ \Big|_0^\pi \tag{5.32}$$

$$- \int_0^\pi \Theta_{q1}(\theta)\ \Theta'_{p2}(\theta)\ d\theta$$

The end point contributions are zero because $\Theta_{q1}(\pi) = 0$, and either $\Theta_{q1}(\theta)$ or $\Theta_{p2}(\theta)$ is odd so that the contribution at $\theta = 0$ is zero. Combining terms, C_q can be written

$$C_q = - \int_0^\pi \Theta_{q1}(\theta)\ \Theta'_{p2}(\theta)\ d\theta \left[\int_{-\pi}^\pi (\Phi_{q1}(\phi)\ \Phi'_{p2}(\phi) \right. \tag{5.33}$$

$$\left. + \Phi'_{q1}(\phi)\ \Phi_{p2}(\phi))\ d\phi \right]$$

The term in the square brackets is just $\Phi_{q1}(\phi)\ \Phi_{p2}(\phi)\Big|_{-\pi}^{\pi}$ which is zero because of the periodicity of the $\Phi(\phi)$ functions. Thus $C_q = 0$ and it follows that $B_q = 0$ for all q.

Now consider

$$A_q = \int_s [\hat{r} \times \overline{M}^{II}_{q2}\ (\overline{R}) \cdot \overline{N}^{I}_{p2}\ (\overline{R}) - \hat{r} \times \overline{M}^{I}_{p2}\ (\overline{R}) \cdot \overline{N}^{II}_{q2}\ (\overline{R})]\ ds \tag{5.34}$$

Again using the auxiliary vector wave functions and manipulating the dot and cross products, A_q can be written

$$A_q = \left[\frac{h^{(2)}_{\nu_{q2}}\ (\kappa r)(r\ j_{\nu_{p2}}\ (\kappa r))' - j_{\nu_{p2}}\ (\kappa r)(r\ h^{(2)}_{\nu_{q2}}\ (\kappa r))'}{\kappa r} \right] \tag{5.35}$$

$$\int_s (\overline{m}_{p2} \cdot \overline{m}_{q2})\ ds$$

Using the definition of $\overline{m}$, the surface integral can be written

$$\int_s (\overline{m}_{p2} \cdot \overline{m}_{q2})\, ds = r^2 \int_s \left[\frac{\Theta'_{p2}(\theta)\ \Phi_{p2}(\phi)\ \Theta'_{q2}(\theta)\ \Phi_{q2}(\phi)}{(h_\theta)^2} \right.$$

$$\left. + \frac{\Theta_{p2}(\theta)\ \Phi'_{p2}(\phi)\ \Theta_{q2}(\theta)\ \Phi'_{q2}(\phi)}{(h_\phi)^2} \right] ds \tag{5.36}$$

where the metric coefficients have been used for compactness. In order to evaluate equation (5.36) a slight digression is necessary. Consider the following vector identity.

$$\nabla \cdot (\psi_{p2} \nabla \psi_{q2}) = \psi_{p2} \nabla^2 \psi_{q2} + \nabla \psi_{p2} \cdot \nabla \psi_{q2} \tag{5.37}$$

Integrate throughout a volume like region I in Figure 7, use Gauss' theorem and $\nabla^2 \psi_{q2} = - \kappa^2 \psi_{q2}$ to obtain

$$\int_s \psi_{p2} \nabla \psi_{q2} \cdot \hat{n} ds = - \kappa^2 \int_v \psi_{q2} \psi_{p2}\, dv + \int_v \nabla \psi_{p2} \cdot \nabla \psi_{q2}\, dv \tag{5.38}$$

Expanding the functions into component form, the left hand side can be written

$$\int_s \psi_{p2} \nabla \psi_{q2} \cdot \hat{n}\, ds$$

$$= R_{p2}(r)\ R'_{q2}(r) \int_s \Theta_{p2}(\theta)\ \Phi_{p2}(\phi)\ \Theta_{q2}(\theta)\ \Phi_{q2}(\phi)\, ds \tag{5.39}$$

where the contribution to the integral along the quarter plane is zero and the surface S is now spherical. The surface integral is the orthogonality relation for the Lamé functions so it is zero unless $q = p$, and the left hand side of equation (5.38) is zero unless $q = p$. The first term on the right hand side of equation (5.38) can be written

$$-\kappa^2 \int_v \psi_{q2} \psi_{p2} \, dv = \int_r \left(\int_s \psi_{q2} \psi_{p2} \, ds \right) dr \tag{5.40}$$

The reason the volume integral can be separated in this way is that

$$dv = h_r \, d_r \, (h_\theta \, h_\phi \, d\theta \, d\phi) = h_r \, dr \, ds = dr \, ds \tag{5.41}$$

The metric coefficient h_r is not a function of θ or ϕ so the surface integral can be integrated independently of r. But the surface integral is the orthogonality relationship again so it is zero unless q = p, and thus the first term on the right hand side of equation (5.38) is zero unless q = p. The conclusion to be drawn at this point is that the gradient of the Lamé functions possess an orthogonality relationship over a volume, i.e.,

$$\int_v \nabla \psi_{p2} \cdot \nabla \psi_{q2} \, dv = 0 \quad q \neq p \tag{5.42}$$

Again using the condition on the metric coefficients, this can be written

$$\int \nabla \psi_{p2} \cdot \nabla \psi_{q2} \, dv = \int_r \left(\int_s \nabla \psi_{p2} \cdot \nabla \psi_{q2} \, ds \right) dr \tag{5.43}$$

For q ≠ p, the volume integral is zero, independently of the radius of the spherical volume. This implies that the surface integral must be identically zero, and thus the gradient of the Lamé functions also possess an orthogonality relationship on the surface of a sphere, i.e.,

$$\int_s \left(\nabla \psi_{p2} \cdot \nabla \psi_{q2} \right) ds = 0 \quad q \neq p \tag{5.44}$$

Expanding the gradients in component form, this can be written

$$\int_s (\nabla \psi_{p2} \cdot \nabla \psi_{q2})\, ds = R'_{p2}(r)\, R'_{q2}(r) \int_s \Theta_{p2}(\theta)\, \Phi_{p2}(\phi)\, \Theta_{q2}(\theta)\, \Phi_{q2}(\phi)\, ds$$

$$+ R_{p2}(r)\, R_{q2}(r) \int_s \left[\frac{\Theta'_{p2}(\theta)\, \Phi_{p2}(\phi)\, \Theta'_{q2}(\theta)\, \Phi_{q2}(\phi)}{(h_\theta)^2} \right.$$

$$\left. + \frac{\Theta_{p2}(\theta)\, \Phi'_{p2}(\phi)\, \Theta_{q2}(\theta)\, \Phi'_{q2}(\phi)}{(h_\phi)^2} \right] ds \qquad (5.45)$$

Again the first term on the right hand side is the orthogonality relationship for the Lamé functions. Thus it has been shown that for $q \neq p$,

$$\int_s \left[\frac{\Theta'_{p2}(\theta)\, \Phi_{p2}(\phi)\, \Theta'_{q2}(\theta)\, \Phi_{q2}(\phi)}{(h_\theta)^2} + \frac{\Theta_{p2}(\theta)\, \Phi'_{p2}(\phi)\, \Theta_{q2}(\theta)\, \Phi'_{q2}(\phi)}{(h_\phi)^2} \right] ds$$

$$= \frac{1}{r^2} \int_s (\overline{m}_{p2} \cdot \overline{m}_{q2})\, ds = 0 \qquad (5.46)$$

and that the auxiliary vector wave functions $\overline{m}_{q2}$ are orthogonal on the surface of a sphere. Thus in equation (5.35) A_q can be written

$$A_q = \begin{cases} 0 & q \neq p \\ A_p = \left[\dfrac{h^{(2)}_{\nu_{p2}}(\kappa r)(r\, j_{\nu_{p2}}(\kappa r))' - j_{\nu_{p2}}(\kappa r)(r\, h^{(2)}_{\nu_{p2}}(\kappa r))'}{\kappa r} \right] r^2 & \\ \displaystyle\int_s \left[\left(\frac{\Theta'_{p2}(\theta)\, \Phi_{p2}(\phi)}{h_\theta}\right)^2 + \left(\frac{\Theta_{p2}(\theta)\, \Phi'_{p2}(\phi)}{h_\phi}\right)^2 \right] ds & q = p \end{cases} \qquad (5.47)$$

Noting that

$$\frac{h^{(2)}_{\nu_{p2}}(\kappa r)(r\, j_{\nu_{p2}}(\kappa r))' - j_{\nu_{p2}}(\kappa r)(r\, h_{\nu_{p2}}(\kappa r))'}{\kappa r} \tag{5.48}$$

$$= \frac{h^{(2)}_{\nu_{p2}}(\kappa r)\, j'_{\nu_{p2}}(\kappa r) - j_{\nu_{p2}}(\kappa r)\, h^{(2)\prime}_{\nu_{p2}}(\kappa r)}{\kappa} = \frac{j}{(\kappa r)^2}$$

and writing out the metric coefficients, it is seen that

$$A_p = \frac{j}{\kappa^2} \int_{-\pi}^{\pi} \int_0^{\pi} \left[\frac{\sqrt{1 - k^2 \cos^2 \theta}}{\sqrt{1 - k'^2 \cos^2 \phi}} (\Theta'_{p2}(\theta)\, \Phi_{p2}(\phi))^2 \right. \tag{5.49}$$

$$\left. + \frac{\sqrt{1 - k'^2 \cos^2 \phi}}{\sqrt{1 - k^2 \cos^2 \theta}} (\Theta_{p2}(\theta)\, \Phi'_{p2}(\phi))^2 \right] d\theta\, d\phi$$

and thus

$$a_p(\overline{R}') = \frac{j\, \kappa\, \overline{M}^I_{p2}(\overline{R}') \cdot \hat{a}}{\Lambda_{p2}} \tag{5.50}$$

where Λ_{p2} is the surface integral in equation (5.52)

$$\Lambda_{p2} = -\, j\, \kappa^2 A_p \tag{5.51}$$

Repeating the above procedure starting with equation (5.12) and letting $\overline{B} = \overline{N}^I_{p1}(\overline{R})$ instead of $\overline{M}^I_{p2}(\overline{R})$, $b_p(\overline{R}')$ is determined to be

$$b_p(\overline{R}') = \frac{j\, \kappa\, \overline{N}^I_{p1}(\overline{R}') \cdot \hat{a}}{\Lambda_{p1}} \tag{5.52}$$

where Λ_{p1} has the same form as Λ_{p2}, the only difference being that Dirichlet functions are used instead of Neumann. Thus,

$$\overline{G}^{II}(\overline{R},\overline{R}') = j\,\kappa \sum_q \left[\frac{\overline{M}^{II}_{q2}(\overline{R})\,\overline{M}^{I}_{q2}(\overline{R}')\cdot\hat{a}}{\Lambda_{q2}} + \frac{\overline{N}^{II}_{q1}(\overline{R})\,\overline{N}^{I}_{q1}(\overline{R}')\cdot\hat{a}}{\Lambda_{q1}} \right] \tag{5.53}$$

Comparing this with equation (5.8), it is seen that

$$\overline{\overline{\Gamma}}(\overline{R},\overline{R}') = j\,\kappa \sum_q \left[\frac{\overline{M}^{II}_{q2}(\overline{R})\,\overline{M}^{I}_{q2}(\overline{R}')}{\Lambda_{q2}} + \frac{\overline{N}^{II}_{q1}(\overline{R})\,\overline{N}^{I}_{q1}(\overline{R}')}{\Lambda_{q1}} \right] \quad |\overline{R}| \geq |\overline{R}'| \tag{5.54}$$

where, with $\alpha = 1$ or 2,

$$\Lambda_{q\alpha} = \int_{-\pi}^{\pi}\int_{0}^{\pi} \left[\frac{\sqrt{1 - k^2\cos^2\theta}}{\sqrt{1 - k'^2\cos^2\phi}} \left(\Theta'_{q\alpha}(\theta)\,\Phi_{q\alpha}(\phi)\right)^2 + \frac{\sqrt{1 - k'^2\cos^2\phi}}{\sqrt{1 - k^2\cos^2\theta}} \left(\Theta_{q\alpha}(\theta)\,\Phi'_{q\alpha}(\phi)\right)^2 \right] d\theta\, d\phi \tag{5.55}$$

To find $\overline{\overline{\Gamma}}(\overline{R},\overline{R}')$ for $|\overline{R}| \leq |\overline{R}'|$, the same procedure could be repeated, this time integrating throughout a volume external to the surface S, and letting the surface S' be at $\overline{R}'$, where $|\overline{R}'| \geq |\overline{R}|$, or the symmetry property of Green's functions could be used. The result is

$$\overline{\overline{\Gamma}}\,(\overline{R},\overline{R}') = j\,\kappa \sum_{q} \left[\frac{\overline{M}^{I}_{q2}\,(\overline{R})\,\overline{M}^{II}_{q2}\,(\overline{R}')}{\Lambda_{q2}} + \frac{\overline{N}^{I}_{q1}\,(\overline{R})\,\overline{N}^{II}_{q1}\,(\overline{R}')}{\Lambda_{q1}} \right] \quad |\overline{R}| \leq |\overline{R}'| \tag{5.56}$$

A point concerning the normalization constants is worthy of note. It is shown in Appendix A that the $\Lambda_{q\alpha}$ occur in discussing the properties of the two-dimensional Sturm-Liouville type Lamé operator. Using the notation in the Appendix

$$\Lambda_{q\alpha} = \langle\, y_{q\alpha},\ L\, y_{q\alpha} \,\rangle \tag{5.57}$$

But,

$$\frac{\langle\, y_{q\alpha},\ L\, y_{qa} \,\rangle}{\langle\, y_{q\alpha},\ \rho\, y_{q\alpha} \,\rangle} = \lambda_{q\alpha} \tag{5.58}$$

Thus, if the normalization of the Lamé functions had been chosen such that

$$\langle\, y_{q\alpha},\ \rho\, y_{q\alpha} \,\rangle = 1 \tag{5.59}$$

i.e., if the Lamé functions had been orthonormalized, then

$$\Lambda_{q\alpha} = \lambda_{q\alpha} = \nu_{q\alpha}\,(\nu_{q\alpha} + 1) \tag{5.60}$$

From the point of view of the applied mathematician, this would be an acceptable normalization of the Lamé functions. This is especially true for the Lamé polynomials. Much confusion is caused by the lack of a standard when reading the work of various authors. The same problem exists with the Legendre polynomials. They are well defined, however they are not orthonormal. This is the reason for the rather complicated normalization constant found in eigenfunction expansions in spherical coordinates.

The normalization constants, $\Lambda_{q\alpha}$, are given in Table 6. In order to orthonormalize the Lamé functions presented in Tables 1 through 4, it is necessary to multiply each Θ and Φ function by $\left(\frac{\lambda q\alpha}{\Lambda q\alpha} \right)^{1/4}$. These numbers are also given in Table 6.

Table 6 - Normalization Constants

A: Dirichlet Problem

ν	μ	Λ_1	$(\lambda_1/\Lambda_1)^{\frac{1}{4}}$
0.296	0.090	2.940	0.601
1.425	0.915	35.103	0.560
1.000	0.000	8.378	0.699
1.130	- 0.455	14.201	0.642
2.480	2.670	269.941	0.423
2.000	1.500	10.050	0.879
2.290	0.215	20.797	0.776
2.000	- 1.500	10.050	0.879
2.040	- 1.705	194.581	0.423
3.495	5.440	3073.608	0.267
3.000	3.873	19.175	0.889
3.410	1.535	67.639	0.687
3.000	0.000	5.741	1.202
3.145	- 0.825	37.245	0.769
3.000	- 3.873	19.175	0.889
3.010	- 3.940	1012.967	0.330
4.499	9.225	26302.376	0.175
4.000	7.190	48.982	0.799
4.470	3.790	370.670	0.507
4.000	2.190	6.276	1.336
4.280	0.335	39.904	0.867
4.000	- 2.190	6.276	1.336
4.050	- 2.575	153.114	0.605
4.000	- 7.190	48.982	0.799
4.005	- 7.205	4648.698	0.256
5.500	14.013	206778.639	0.115
5.000	11.489	157.321	0.661
5.490	7.110	2517.012	0.345
5.000	5.196	10.424	1.302
5.499	2.110	136.722	0.715
5.000	0.000	4.265	1.629
5.150	- 1.170	62.106	0.845
5.000	- 5.196	10.424	1.302
5.015	- 5.335	970.112	0.420
5.000	-11.489	157.321	0.661
5.000	-11.493	73566.927	0.142

Table 6 - (continued)

ν	μ	Λ_1	$(\lambda_1/\Lambda_1)^{\frac{1}{4}}$
6.500	19.805	164299.188	0.131
6.000	16.783	533.201	0.530
6.499	11.455	11646.480	0.254
6.000	9.114	23.400	1.157
6.465	4.850	1172.142	0 450
6.000	2.832	4.679	1.731
6.282	0.445	1300.687	0.433
6.000	- 2.832	4.679	1.731
6.055	- 3.380	470.909	0.549
6.000	- 9.114	23.400	1.157
6.005	- 9.160	4881.640	0.305
6.000	-16.783	533.201	0.530
6.000	-16.783	427489.535	0.099
7.500	26.597	523205.875	0.105
7.000	23.075	841.478	0.508
7.500	16.820	62918.110	0.178
7.000	14.000	62.554	0.973
7.492	8.695	411996.648	0.111
7.000	6.444	7.341	1.662
7.393	2.660	1053.201	0.493
7.000	0.000	3.448	2.007
7.157	- 1.500	1947.103	0.416
7.000	- 6.444	7.341	1.662
7.020	- 6.660	1614.970	0.432
7.000	-14.000	62.554	0.973
7.000	-14.010	8308.581	0.287
7.000	-23.075	841.478	0.508
7.000	-23.075	992383.008	0.087
8.500	34.389	203797.938	0.141
8.000	30.367	5077.147	0.345
8.500	23.191	115954.960	0.162
8.000	19.876	156.869	0.823
8.497	13.590	866869.250	0.098
8.000	10.949	14.942	1.482
8.463	5.885	2928.812	0.407
8.000	3.441	3.745	2.094
8.275	0.535	464459.938	0.113
8.000	- 3.441	3.745	2.094
8.073	- 4.170	758.318	0.557
8.000	-10.949	14.942	1.482
8.005	-11.015	1130.930	0.502
8.000	-19.876	156.869	0.823
8.000	-19.879	6703.902	0.322
8.000	-30.367	5077.147	0.345
8.000	-30.367	902867.680	0.094

Table 6 - (continued)

ν	μ	Λ_1	$(\lambda_1/\Lambda_1)^{\frac{1}{4}}$
9.500	--	--	--
9.000	38.660	677.440	0.604
9.500	--	--	--
9.000	26.751	645.618	0.611
~9.50	--	--	--
9.000	16.431	37.872	1.242
--	--	--	--
9.000	7.640	5.650	1.998
--	--	--	--
9.000	0.000	2.915	2.357
--	--	--	--
9.000	- 7.640	5.650	1.998
--	--	--	--
9.000	-16.413	37.872	1.242
~9.00	~ -16.4	--	--
9.000	-26.751	645.618	0.611
~9.00	~ -26.8	--	--
9.000	-38.660	677.440	0.604
9.000	-38.660	607247.125	0.110
B: Neumann Problem			
1.000	0.500	16.750	0.588
0.814	- 0.194	12.233	0.589
1.000	- 0.500	16.750	0.588
2.000	1.732	120.627	0.472
1.595	0.795	9.843	0.805
2.000	0.000	20.094	0.739
1.955	- 1.552	12.890	0.818
2.000	- 1.737	120.627	0.472
3.000	3.950	1058.871	0.326
2.520	2.625	14.021	0.892
3.000	0.950	45.070	0.718
2.803	- 0.350	6.998	0.111
3.000	- 0.950	45.070	0.718
2.990	- 3.890	22.573	0.853
3.000	- 3.950	1058.871	0.326
4.000	7.211	9332.023	0.215
3.505	5.430	30.317	0.850
4.000	2.646	178.674	0.578
3.617	1.260	6.567	1.263
4.000	0.000	40.897	0.836

Table 6 - (continued)

ν	μ	Λ_1	$(\lambda_1/\Lambda_1)^{\frac{1}{4}}$
3.940	- 2.315	6.817	1.300
4.000	- 2.646	178.674	0.578
3.998	- 7.195	56.527	0.771
4.000	- 7.211	9332.023	0.215
5.000	11.494	76184.075	0.141
4.500	9.218	85.544	0.733
5.000	5.363	1047.489	0.411
4.530	3.680	8.176	1.323
5.000	1.369	73.363	0.800
4.795	- 0.495	5.281	1.515
5.000	- 1.369	73.363	0.800
4.984	- 5.230	10.556	1.296
5.000	- 5.363	1047.489	0.411
5.000	-11.491	176.504	0.642
5.000	-11.494	76184.075	0.141
6.000	16.784	617657.477	0.090
5.500	14.012	284.757	0.595
6.000	9.165	6926.720	0.279
5.507	7.060	15.340	1.236
6.000	3.507	236.801	0.649
5.627	1.692	4.679	1.680
6.000	0.000	62.880	0.904
5.927	- 3.015	4.486	1.739
6.000	- 3.507	236.801	0.649
5.997	- 9.120	26.646	1.120
6.000	- 9.165	6926.720	0.279
6.000	-16.783	637.161	0.507
6.000	-16.784	617657.477	0.090
7.000	23.076	70189.938	0.168
6.500	19.804	1031.706	0.466
7.000	14.014	46539.142	0.186
6.501	11.440	69.934	0.914
7.000	6.708	1135.994	0.471
6.540	4.675	5.719	1.714
7.000	1.770	102.166	0.860
6.792	- 0.630	169.150	0.748
7.000	- 1.770	102.166	0.860
6.980	- 6.550	4.261	1.901
7.000	- 6.708	1135.994	0.471
7.000	-14.002	188.385	0.738
7.000	-14.014	46539.142	0.186
7.000	-23.075	2412.174	0.390
7.000	-23.076	701895.938	0.094

Table 6 - (continued)

ν	μ	Λ_1	$(\lambda_1/\Lambda_1)^{\frac{1}{4}}$
8.000	30.368	943368.000	0.093
7.500	26.597	3927.403	0.357
8.000	19.880	294906.848	0.125
7.500	16.816	4.848	1.904
8.000	11.036	6352.965	0.326
7.510	8.625	9.765	1.599
8.000	4.334	292.966	0.704
7.632	2.095	94.074	0.915
8.000	0.000	85.737	0.957
7.923	- 3.685	6.299	1.830
8.000	- 4.334	292.966	0.704
7.995	-10.965	5.347	1.915
8.000	-11.036	6352.965	0.326
8.000	-19.877	819.559	0.544
8.000	-19.880	294906.848	0.125
8.000	-30.368	14267.483	0.267
8.000	-30.368	943368.000	0.093
9.000	38.660	221153.000	0.142
8.500	34.389	11763.660	0.288
9.000	26.752	510399.188	0.115
8.500	23.190	462.985	0.646
9.000	16.438	36518.688	0.223
8.503	13.570	36.122	1.223
9.000	8.004	1233.913	0.520
8.545	5.630	36.915	1.219
9.000	2.158	131.879	0.909
8.787	- 0.770	2369.904	0.436
9.000	- 2.158	131.879	0.909
8.980	- 7.710	8.552	1.799
9.000	- 8.004	1233.931	0.520
9.000	-16.420	9.861	1.738
9.000	-16.438	36518.688	0.223
9.000	-26.751	86.694	1.009
9.000	-26.752	510399.188	0.115
9.000	-38.660	78706.980	0.184
9.000	-38.660	221153.000	0.142

CHAPTER VI

DISCUSSION OF THE FIELDS AND CURRENTS FOR SOME SPECIAL CASES

This chapter contains several numerical examples of the exact solution given in Chapter V. First, the exact fields and current for an infinitesimal dipole source are derived. These are then evaluated in the vicinity of the tip of the quarter plane. Several source locations and orientations are considered. The dominant behavior near the tip of a wide angle sector and a narrow angle sector are also discussed. The reciprocal case of a source close to the tip of the quarter plane is also considered, and several far field patterns are given.

Fields and Currents for a Unit Dipole Source

The dyadic Green's functions derived in Chapter V are to be used in equation (5.3) in order to determine the vector $\overline{E}$ field at $\overline{R}$ due to a current source at $\overline{R}'$. If the source is a unit dipole source at $\overline{R}_o$, i.e., if

$$j\omega\mu \, \overline{J} \, (\overline{R}') = \hat{a} \, \delta \, (\overline{R}'-\overline{R}_o) \tag{6.1}$$

where $\hat{a}$ is just a unit vector at $\overline{R}_o$, the $\overline{E}$ field is simply

$$\overline{E}(\overline{R}) = \begin{cases} j\kappa \sum_q \left[\overline{M}^{II}_{q2}(\overline{R}) \left(\dfrac{\overline{M}^{I}_{q2}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{q2}} \right) + \overline{N}^{II}_{q1}(\overline{R}) \left(\dfrac{\overline{N}^{I}_{q1}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{q1}} \right) \right] & r \geq r_o \qquad (6.2a) \\[2ex] j\kappa \sum_q \left[\overline{M}^{I}_{q2}(\overline{R}) \left(\dfrac{\overline{M}^{II}_{q2}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{q2}} \right) + \overline{N}^{I}_{q1}(\overline{R}) \left(\dfrac{\overline{N}^{II}_{q1}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{q1}} \right) \right] & r \leq r_o \qquad (6.2b) \end{cases}$$

The $\overline{H}$ field is easily determined using Maxwell's equations, and the properties of the $\overline{M}$ and $\overline{N}$ vectors given by equations (4.10) - (4.13).

$$\overline{H}(\overline{R}) = \frac{j}{\omega\mu} \nabla \times \overline{E}(\overline{R}) = \begin{cases} -\kappa Y_o \sum_q \left[\overline{N}^{II}_{q2}(\overline{R}) \left(\dfrac{\overline{M}^{I}_{q2}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{q2}} \right) + \overline{M}^{II}_{q1}(\overline{R}) \left(\dfrac{\overline{N}^{I}_{q1}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{q1}} \right) \right] & r \geq r_o \qquad (6.3a) \\[2ex] -\kappa Y_o \sum_q \left[\overline{N}^{I}_{q2}(\overline{R}) \left(\dfrac{\overline{M}^{II}_{q2}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{q2}} \right) + \overline{M}^{I}_{q1}(\overline{R}) \left(\dfrac{\overline{N}^{II}_{q1}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{q1}} \right) \right] & r \leq r_o \qquad (6.3b) \end{cases}$$

Y_o is the admittance of the medium,

$$Y_o = \sqrt{\frac{\varepsilon}{\mu}} \; . \tag{6.4}$$

The current on the quarter plane can be found by evaluating the tangential $\overline{H}$ field at the quarter plane. In order to do this it is necessary to consider the even and odd parts of the $\overline{H}$ field. Equation (6.3a) is

$$\overline{H}(\overline{R}) = -\kappa Y_o \sum_{\ell} \Bigg[\overline{N}^{II}_{e\ell 2}(\overline{R}) \left(\frac{\overline{M}^{I}_{e\ell 2}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{e\ell 2}} \right) + \overline{N}^{II}_{o\ell 2}(\overline{R}) \left(\frac{\overline{M}^{I}_{o\ell 2}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{o\ell 2}} \right) + \overline{M}^{II}_{e\ell 1}(\overline{R}) \left(\frac{\overline{N}^{I}_{e\ell 1}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{e\ell 1}} \right) + \overline{M}^{II}_{o\ell 1}(\overline{R}) \left(\frac{\overline{N}^{I}_{o\ell 1}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{o\ell 1}} \right) \Bigg] \tag{6.5}$$

Recalling the notation that was introduced in Chapter IV, the subscript q is used to simplify the group of subscripts ${}^{e}_{o}\ell$. Thus if it is necessary to identify even and odd functions, the shorthand notation q can not be used and it is necessary to use the more bulky notation in equation (6.5).

The surface current on the quarter plane is

$$\overline{J}(\overline{R}) = \hat{n} \times [\overline{H}(r, \theta=\pi, 0 \leq \phi \leq \pi) - \overline{H}(r, \theta=\pi, \pi \leq \phi \leq 2\pi)] \tag{6.6}$$

where $\hat{n}$ is a unit normal vector and $\hat{n} = \hat{y}$.

Performing the calculations, it is found that $\overline{N}^{II}_{e\ell 2}(\overline{R})$ and $\overline{M}^{II}_{o\ell 1}(\overline{R})$ do not contribute to the current. The actual calculation is very complicated because of the different unit vector directions on opposite sides of the quarter plane, and because a point is described by $(r,\ \theta = \pi,\ \phi)$ on one side of the quarter plane and by $(r,\ \theta = \pi,\ 2\pi - \phi)$ on the other side. The result, however, is obvious after one realizes that $\overline{N}^{II}_{e\ell 2}(\overline{R})$ and $\overline{M}^{II}_{o\ell 1}(\overline{R})$ are composed of Lamé polynomials. This implies that these vector wave functions are free space solutions of the vector wave equation (no scattering body) and are therefore continuous at the quarter plane. The contribution of the remaining two terms to the surface current is

$$\overline{J}(\overline{R}) = -2\kappa\, Y_o \sum_{\ell} \Bigg[\frac{\overline{M}^{I}_{o\ell 2}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{o\ell 2}}\, \Theta_{c\ell 2}(\pi) \Bigg(\frac{-\nu(\nu+1)\, h^{(2)}_{\nu}(\kappa r)}{\kappa r}\, \Phi_{o\ell 2}(\phi)\, \hat{\phi} + \frac{(r\, h^{(2)}_{\nu}(\kappa r))'}{\kappa r}\, \frac{\sqrt{1 + \sin^2 \phi}\ \ \Phi'_{c\ell 2}(\phi)\, \hat{r}}{\sin \phi} \Bigg)$$

$$+ \frac{\overline{N}^{I}_{e\ell 1}(\overline{R}_c) \cdot \hat{a}}{\Lambda_{e\ell 1}}\, \Theta'_{e\ell 1}(\pi)\, \frac{h^{(2)}_{\nu}(\kappa r)}{\kappa r}\, \frac{\Phi_{e\ell 1}(\phi)\, \hat{r}}{\sin \phi} \Bigg] \quad r \geq r_c \qquad (6.7)$$

For the sake of notational economy, the eigenvalue ν has not been subscripted. This will occur whenever it is clear that the eigenvalue to be used corresponds to an identified eigenfunction.

Equation (6.7) was derived from equation (6.3a) and is thus good for $r \geq r_c$. For $r \leq r_c$, the surface current is

$$\overline{J}(\overline{R}) = -2\kappa\, Y_c \sum_{\ell} \left[\frac{\overline{M}^{II}_{o\ell 2}(\overline{R}_c) \cdot \hat{a}}{\Lambda_{o\ell 2}} \; \Theta_{c\ell 2}(\pi)\,(-\nu(\nu+1)\; \frac{j_\nu(\kappa r)}{\kappa r} \; \Phi_{c\ell 2}(\phi)\; \hat{\phi} \right.$$

$$+ \frac{(r\, j_\nu\, (\kappa r))'}{\kappa r} \; \frac{\sqrt{1 + \sin^2 \phi}}{\sin \phi} \; \Phi'_{c\ell 2}(\phi)\; \hat{r}\;)$$

$$\left. + \frac{\overline{N}^{II}_{e\ell 1}(\overline{R}_o) \cdot \hat{a}}{\Lambda_{e\ell 1}} \; \frac{\Theta'_{e\ell 1}(\pi)\; j_\nu(\kappa r)}{\sin \phi} \; \Phi_{e\ell 1}(\phi)\; \hat{r} \right] \quad r \le r_c \qquad (6.8)$$

The variable ϕ in equation (6.7) and (6.8) is restricted such that $0 \le \phi \le \pi$.

Up to this point, no approximations have been made. In principle, equations (6.2), (6.3), (6.7) and (6.8) for $\overline{E}$, $\overline{H}$, and $\overline{J}$ can be used to determine the exact fields and currents anywhere due to a dipole source with any location and any orientation. Admittedly the equations are complicated but it is practicable to use them in calculations where the source point or observation point is no more than one or two wavelengths from the tip. Moreover, it may be possible to develop asymptotic expressions starting with these expansions.

Fields and Currents Near the Tip of the Quarter Plane

Consider what happens as κr becomes very small. The $\overline{E}$ field for $r_o > r$ is given by equation (6.2b). The $\overline{R}$ dependent terms of the $\overline{E}$ field are $\overline{M}^{I}_{q2}(\overline{R})$ and $\overline{N}^{I}_{q1}(\overline{R})$. First, consider $\overline{M}^{I}_{q2}(\overline{R})$. From equation (4.17)

$$\overline{M}^{I}_{q2}(\overline{R}) = j_\nu\; (\kappa r)\; \overline{m}_{q2} \qquad (6.9)$$

For small κr, the Bessel function can be written

$$j_\nu\ (\kappa r) \approx \frac{\sqrt{\pi}\ (\kappa r)^\nu}{2^{\nu+1}\ \Gamma(\nu+1.5)} \tag{6.10}$$

where $\Gamma(x)$ is the Gamma function. The lowest Neumann eigenvalue is $\nu = 0.814$. For this eigenvalue

$$\overline{M}^I_{o2}(\overline{R}) \approx 0.428\ (\kappa r)^{0.814}\ \ \overline{m}_{o2}\ (\nu = 0.814) \tag{6.11}$$

Note that the subscript ℓ has been suppressed. This will occur in all of the following equations when it is clearly understood which eigenvalue is being used. The next highest eigenvalue is $\nu = 1$. The two corresponding Lamé functions are even, and the vector wave functions are

$$\overline{M}^I_{e2}(\overline{R}) \approx \frac{\kappa r}{3} \begin{cases} \overline{m}_{e2}\ (\nu = 1,\ \mu = 0.5) \\ \overline{m}_{e2}\ (\nu = 1,\ \mu = -0.5) \end{cases} \tag{6.12}$$

Next, $\overline{N}_{q1}(\overline{R})$ is examined when κr is small. From equation (4.18)

$$\overline{N}^I_{q1}(\overline{R}) = \frac{j_\nu\ (\kappa r)}{\kappa r}\ \overline{\ell}_{q1} + \frac{(r\ j_\nu\ (\kappa r))'}{\kappa r}\ (\hat{r} \times \overline{m}_{q1}) \tag{6.13}$$

For small κr

$$\frac{(r\ j_\nu\ (\kappa r))'}{\kappa r} \approx \frac{\sqrt{\pi}\ (\nu+1)(\kappa r)^{\nu-1}}{2^\nu\ (2\nu+1)\Gamma(\nu+0.5)} \tag{6.14}$$

The lowest Dirichlet eigenvalue is $\nu = 0.296$. The corresponding Lamé function is even, and the vector wave function is

$$\begin{aligned} \overline{N}^I_{e1}(\overline{R}) \approx 0.776\ (\kappa r)^{-0.704}\ \ &[\ell_{e1}(\nu = 0.296) \\ &+ 1.296\ (\hat{r} \times \overline{m}_{e1}\ (\nu = 0.296))\ ] \end{aligned} \tag{6.15}$$

The next eigenvalue is $\nu = 1$. The corresponding Lamé function is odd, and the vector wave function is

$$\overline{N}^{I}_{o1}(\overline{R}) \approx \frac{1}{3} \left[\overline{\ell}_{c1} (\nu = 1) + 2 (\hat{r} \times \overline{m}_{c1} (\nu = 1))\right] \tag{6.16}$$

The dominant term for κr very small is obviously $\overline{N}^{I}_{e1}(\overline{R})$. The normalization constant associated with this term is (from Table 6)

$$\Lambda_{e1} = 2.940 \tag{6.17}$$

Thus the $\overline{E}$ field very close to the tip of the quarter plane can be written

$$\overline{E}(\overline{R}) = j \kappa (0.264)(\overline{N}^{II}_{e1} (\overline{R}_o) \cdot \hat{a})(\kappa r)^{-0.704}$$

$$\left[\overline{\ell}_{e1} + 1.296 (\hat{r} \times \overline{m}_{e1}) \right] \quad \kappa r << 1 \tag{6.18}$$

where it is understood that the eigenfunctions to be used in the vector wave functions correspond to the eigenvalue $\nu = 0.296$.

Next consider the $\overline{H}$ field. Using the small argument approximation for the Bessel functions, and evaluating the first few terms, it is seen that the dominant behavior of the $\overline{H}$ field near the tip is

$$\overline{H}(\overline{R}) = -\kappa Y_o (0.024)(\overline{M}^{II}_{c2}(\overline{R}_o) \cdot \hat{a})(\kappa r)^{-0.186}$$

$$\left[\overline{\ell}_{c2} + 1.814 (\hat{r} \times \overline{m}_{c2}) \right] \quad \kappa r << 1 \tag{6.19}$$

The eigenfunctions in this equation correspond to the eigenvalue $\nu = 0.814$. The normalization constant is

$$\Lambda_{o2} = 12.233 \tag{6.20}$$

The dominant term in the surface current is the same as for $\overline{H}$ and is given by

$$\overline{J}(R) = -2\ \kappa\ Y_o\ (0.031)(\overline{M}^{II}_{o2}(\overline{R}_o)\cdot \hat{a})(\kappa r)^{-0.186}$$

$$\left[-\Phi_{o2}(\phi)\ \hat{\phi} + 1.228\ \frac{\sqrt{1+\sin^2\phi}}{\sin\phi}\ \Phi'_{o2}(\phi)\ \hat{r}\right] \kappa r << 1 \qquad (6.21)$$

Equations (6.18), (6.19), and (6.21) give only the singular components of the fields and current near the tip of the quarter plane. These equations govern the dominant field and current behavior whenever they do not vanish. For certain source locations and orientations $\hat{a}\cdot\overline{N}^{II}_{e1}$ and $\hat{a}\cdot\overline{M}^{II}_{o2}$ will be zero and it will be necessary to investigate other terms in the field and current expansions. This will be done later.

Next, the singular fields are examined at the edges and on the surface of the quarter plane. From equation (6.18), the $\overline{E}$ field, for a constant value of r is

$$\overline{E}(\overline{R}) = A\ \Theta_{e1}(\theta)\ \Phi_{e1}(\phi)\ \hat{r}$$

$$+ \frac{B\sqrt{1+\sin^2\phi}}{\sqrt{\sin^2\theta+\sin^2\phi}}\ \Theta_{e1}(\theta)\ \Phi'_{e1}(\phi)\ \hat{\phi}$$

$$+ \frac{B\sqrt{1+\sin^2\theta}}{\sqrt{\sin^2\theta+\sin^2\phi}}\ \Theta'_{e1}(\theta)\ \Phi_{e1}(\phi)\ \hat{\theta} \qquad \kappa r << 1 \qquad (6.22)$$

where A and B are constants. The edges of the quarter plane correspond to ($\theta = \pi$, $\phi = 0$) and ($\theta = \pi$, $\phi = \pi$). Approaching the edge on the $\phi = 0$ sector (see Figure 8), it is seen that the $\hat{r}$ component goes to zero, the $\hat{\theta}$ component is zero, and the $\hat{\phi}$ component goes to infinity. Similar behavior is found for the edge ($\theta = \pi$, $\phi = \pi$). Approaching the edges from the quarter plane itself ($\theta = \pi$), it is seen that the $\hat{r}$ component is zero, the $\hat{\phi}$ component is zero, and the $\hat{\theta}$ component goes to infinity (see Figure 9). This edge behavior is consistent with the results from half-plane theory. On the quarter plane itself, the parallel ($\hat{r}$, $\hat{\phi}$) components are zero, and the perpendicular ($\hat{\theta}$) component is finite except at the edge.

From equation (6.19) the $\overline{H}$ field is

$$\overline{H}(\overline{R}) = C\, \Theta_{c2}(\theta)\, \Phi_{o2}(\phi)\, \hat{r}$$

$$+ \frac{D\sqrt{1 + \sin^2 \phi}}{\sqrt{\sin^2 \theta + \sin^2 \phi'}} \quad \Theta_{o2}(\theta)\, \Phi'_{c2}(\phi)\, \hat{\phi}$$

$$+ \frac{D\sqrt{1 + \sin^2 \theta}}{\sqrt{\sin^2 \theta + \sin^2 \phi}} \quad \Theta'_{c2}(\theta)\, \Phi_{o2}(\phi)\, \hat{\theta}$$

$$\kappa r << 1 \qquad (6.23)$$

where C and D are constants. Approaching the ($\theta = \pi$, $\phi = 0$) edge from the $\phi = 0$ sector, it is seen that the $\hat{r}$ component is zero, the $\hat{\phi}$ component goes to infinity, and the $\hat{\theta}$ component is zero. Similar behavior is found for the other edge. Approaching the edges from the quarter plane itself ($\theta = \pi$), it is seen that the $\hat{r}$ component goes to zero, the $\hat{\phi}$ component goes to infinity, and the $\hat{\theta}$ component is zero. This edge behavior for the $\overline{H}$ field is also consistent with half-plane theory. On the quarter plane itself, the parallel components ($\hat{r}$, $\hat{\phi}$)

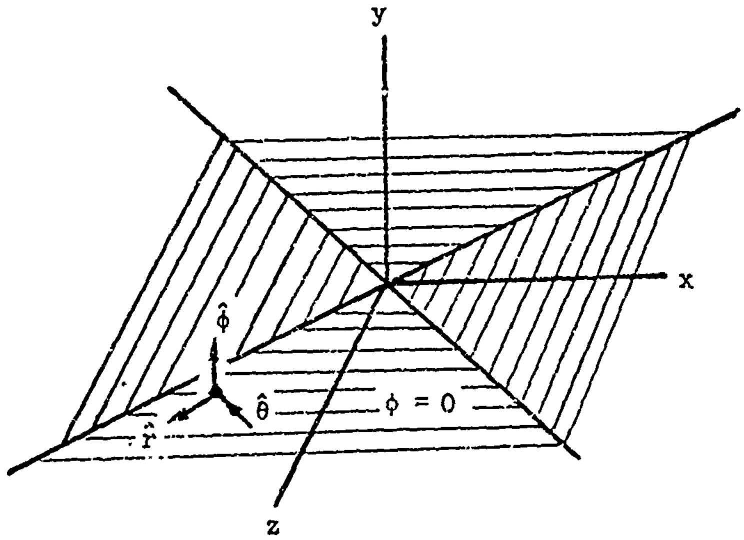

Fig. 8. Unit vectors on the ϕ=0 sector

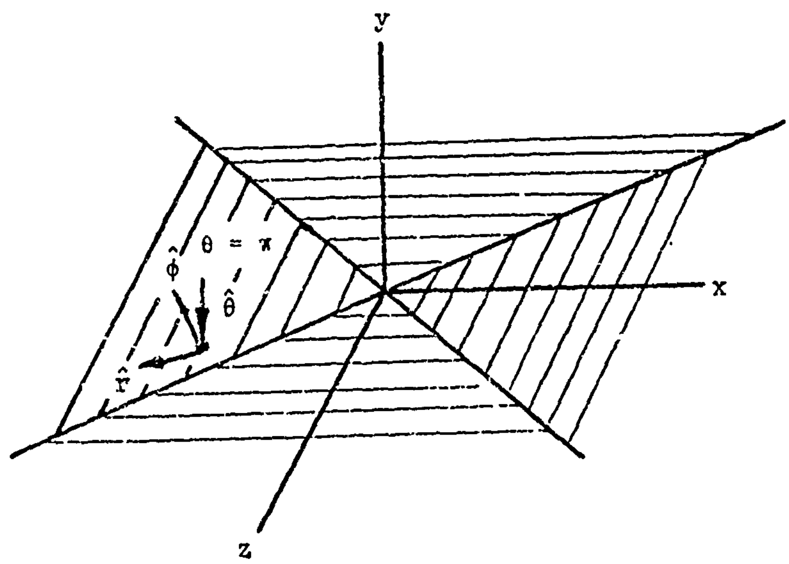

Fig. 9. Unit vectors on the θ=π sector

are finite except at the edges, where the $\hat{r}$ component is zero, and the $\hat{\phi}$ component is infinite. The perpendicular component $(\hat{\theta})$ is zero.

It was mentioned in Chapters IV and V that the electromagnetic fields must satisfy edge and tip conditions. These conditions are necessary for uniqueness, and they do not occur in the case of scattering bodies with well defined surface normals. A good summary of this subject can be found in Jones [4,19]. Jones describes the conditions imposed by several authors and concludes that although their viewpoints are different, their results are in agreement. For the quarter plane problem, perhaps the simplest way of stating the condition is to require that no energy be radiated from the tip or edges. Thus the condition can be reduced to satisfying

$$\mathrm{Re} \int_S \overline{E} \times \overline{H}^* \cdot \overline{ds} = 0$$

on a surface enclosing the tip (or edge) as this surface shrinks to the tip (or edge). This can be done by inspecting the order of the singularities of the proper field components as the edges and tip are approached. The tip and edge conditions are satisfied by the solutions given here.

From equation (6.21), the surface current on the quarter plane is

$$\overline{J}(\overline{R}) = E\, \Phi_{02}(\phi)\, \hat{\phi} + F \frac{\sqrt{1 + \sin^2 \phi}}{\sin \phi}\, \Phi'_{02}(\phi)\, \hat{r} \qquad \kappa r << 1 \qquad (6.24)$$

Figures 10, 11, and 12 are sketches of the $\hat{r}$ and $\hat{\phi}$ components of the surface current, and the total surface current flow near the tip of the quarter plane.

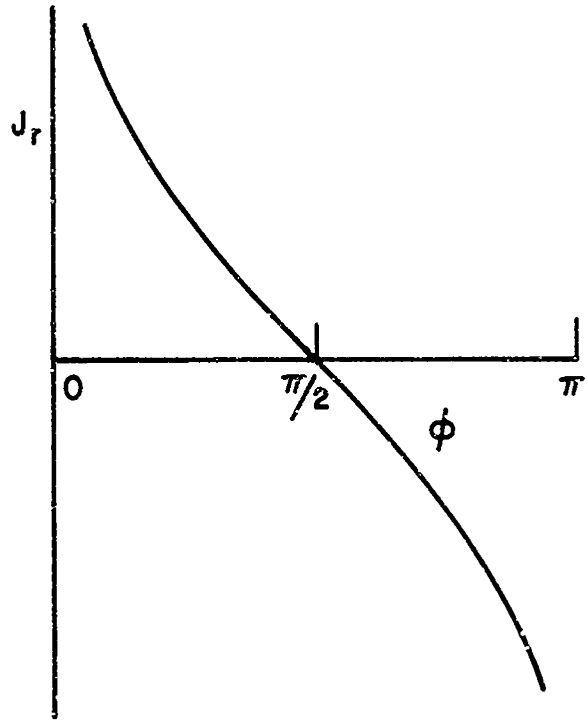

Fig. 10. Variation of J_r for $\kappa r \ll 1$
(From equation 6.24)

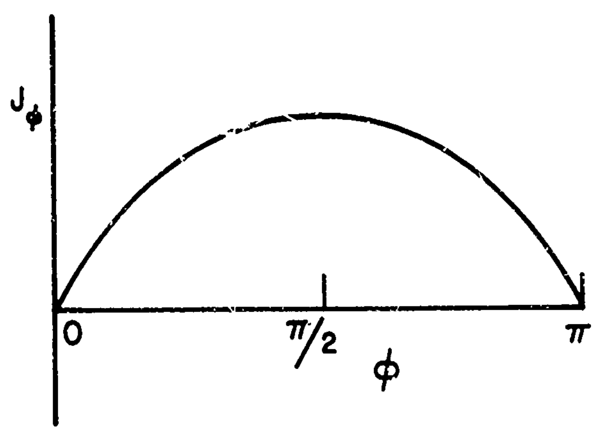

Fig. 11. Variation of J_ϕ for $\kappa r \ll 1$
(From equation 6.24)

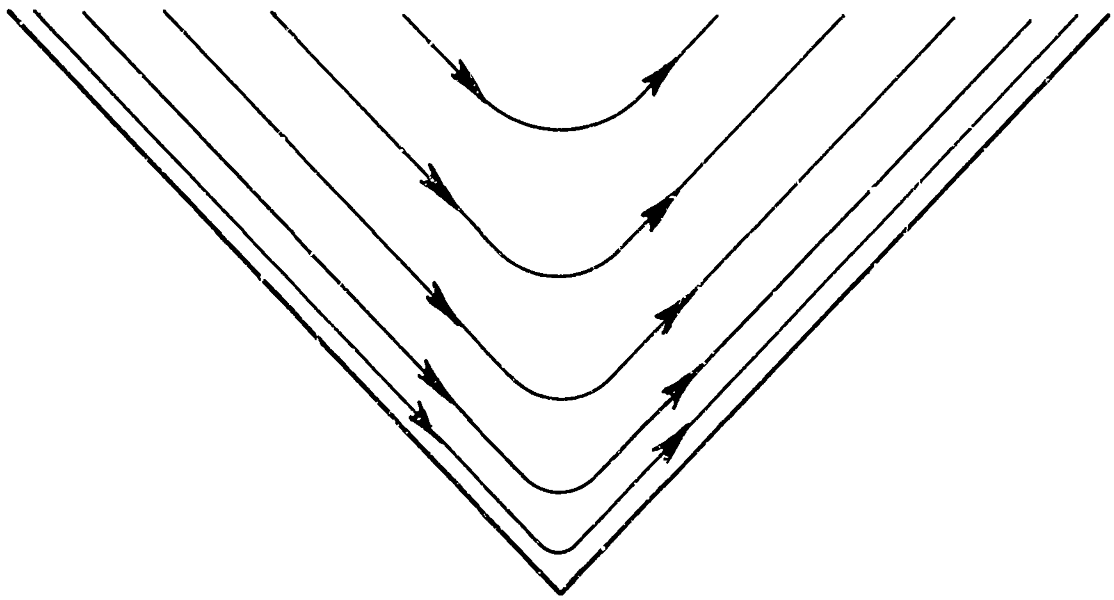

Fig. 12. Surface current flow near the tip of the quarter plane (From equation 6.24)

All of the fields and currents just described are the dominant terms whenever they are not zero. In order to determine when these terms vanish, it is necessary to examine the source terms. The source term for the E field is

$$\overline{N}^{II}_{e1}(\overline{R}_o) \cdot \hat{a} = \frac{0.384\, h^{(2)}_{0.296}(\kappa r_o)}{\kappa r_o} \Theta_{e1}(\theta_o)\, \Phi_{e1}(\phi_o) \hat{r}_o \cdot \hat{a}$$

$$+ \frac{(r_o\, h^{(2)}_{0.296}(\kappa r_o))'}{\kappa r_o \sqrt{\sin^2\theta_o + \sin^2\phi_o}}$$

$$\left(\sqrt{1+\sin^2\phi_o}\;\Theta_{e1}(\theta_o)\, \Phi'_{e1}(\phi_o)\, \hat{\phi}_o \cdot \hat{a} \right.$$

$$\left. + \sqrt{1+\sin^2\theta_o}\;\Theta'_{e1}(\theta_o)\, \Phi_{e1}(\phi_o)\, \hat{\theta}_o \cdot \hat{a} \right) \qquad (6.25)$$

First, consider a source in the $\hat{r}_o$ direction, i.e., $\hat{a} = \hat{r}_o$. Then

$$\overline{N}^{II}_{e1}(\overline{R}_o) \cdot \hat{a} = \frac{(0.384)\, h^{(2)}_{0.296}(\kappa r_o)\, \Theta_{e1}(\theta_o)\, \Phi_{e1}(\phi_o)}{\kappa r_o} \qquad (6.26)$$

In general, this term is not zero for any θ_o or ϕ_o except of course $\theta_o = \pi$. Next consider a source in the $\hat{\phi}_o$ direction. Then

$$\overline{N}^{II}_{e1}(\overline{R}_o) \cdot \hat{a} = \frac{(r_o\, h_{0.296}(\kappa r_o))'}{\kappa r_o} \; \frac{\sqrt{1+\sin^2\phi_o}\;\Theta_{e1}(\theta_o)\, \Phi'_{e1}(\phi_o)}{\sqrt{\sin^2\theta_o + \sin^2\phi_o}} \qquad (6.27)$$

The eigenfunction $\Phi_{e1}(\phi_o)$ is of the form

$$\Phi_{e1}(\phi_o) = \sum_{m=0}^{\infty} B_{2m} \cos 2m\,\phi_o$$

so that

$$\Phi'_{e1}(\phi_o) = -\sum_{m=1}^{\infty} 2m\, B_{2m} \sin 2m\, \phi_o$$

Thus this term is zero for $\phi_o = \pi/2,\ 3\pi/2$, which corresponds to the $z = 0$ plane. The vector $\hat{a} = \hat{\phi}_o$ is perpendicular to this plane. These results imply that equation (6.18) for the $\overline{E}$ field close to the tip of the quarter plane is not valid for this particular source location and orientation. Equation (6.27) is also zero for $\phi_o = 0, \pi$. A source in the $\hat{\phi}_o$ direction at $\phi_o = 0, \pi$ is a source in the plane of the quarter plane ($y = 0$ plane) but not on the quarter plane itself, and oriented perpendicular to it. It is not surprising that the current vanishes for this case, since this type of source produces no scattered field. It is also zero for $\theta_o = \pi$. Next consider a source in the $\hat{\theta}_o$ direction. Then

$$\overline{N}^{II}_{e1}(\overline{R}_o) \cdot \hat{a} = \frac{(r_o\, h^{(2)}_{0.296}\,(\kappa r_o))'}{\kappa r_o}\ \frac{\sqrt{1 + \sin^2\theta_o}\ \ \Theta'_{e1}(\theta_o)\ \Phi_{e1}(\phi_o)}{\sqrt{\sin^2\theta_o + \sin^2\phi_o}} \tag{6.28}$$

This term is zero for $\theta_o = 0$. Again this corresponds to a source located in the plane of the quarter plane and oriented perpendicular to it; on the other hand, equation (6.28) is not zero for a source on the quarter plane itself.

Since equation (6.18) does not describe the dominant $\overline{E}$ field when the source is in the $z = 0$ plane and oriented perpendicular to it, it is necessary to look for the next most dominant term. This term corresponds to $\overline{N}^{I}_{e1}(\overline{R})$ with $\nu = 1.130$. The $\overline{E}$ field is

$$\overline{E}(\overline{R})=j\kappa(0.011)\,(\overline{N}^{II}_{e1}(\overline{R}_o)\cdot\hat{a})(\kappa r)^{0.13}\,[\overline{\ell}_{e1}+2.13\,(\hat{r}\times\overline{m}_{e1})]$$

$$\kappa r << 1 \qquad (6.29)$$

The normalization constant is

$$\Lambda_{e1} = 14.201 \qquad (6.30)$$

Note that this field is not singular at the tip. The edge behavior, however, is the same as before.

The source term for the dominant $\overline{H}$ field and current is

$$\overline{M}^{II}_{o2}(\overline{R}_o)\cdot\hat{a} = \frac{h^{(2)}_{0.814}(\kappa r_o)}{\sqrt{\sin^2\theta_o+\sin^2\phi_o}}$$

$$[\sqrt{1+\sin^2\theta_o}\;\Theta'_{o2}(\theta_o)\,\Phi_{o2}(\phi_o)\,\hat{\phi}_o\cdot\hat{a}$$

$$-\sqrt{1+\sin^2\phi_o}\;\Theta_{o2}(\theta_o)\,\Phi'_{o2}(\phi_o)\,\hat{\theta}_o\cdot\hat{a}\,] \qquad (6.31)$$

This term is, of course, zero for a source in the $\hat{r}_o$ direction. For a source in the $\hat{\phi}_o$ direction,

$$\overline{M}^{II}_{o2}(\overline{R}_o)\cdot\hat{a} = h^{(2)}_{0.814}(\kappa r_o)\,\frac{\sqrt{1+\sin^2\theta_o}\;\Theta'_{o2}(\theta_o)\,\Phi_{o2}(\phi_o)}{\sqrt{\sin^2\theta_o+\sin^2\phi_o}} \qquad (6.32)$$

This term is not zero, in general, for any source location except $\phi_o = 0$, which has already been discussed, and of course for $\theta_o = \pi$. For a source in the $\hat{\theta}_o$ direction,

$$\overline{M}^{II}_{o2}(\overline{R}_o)\cdot\hat{a} = -\,h^{(2)}_{0.814}(\kappa r_o)\,\frac{\sqrt{1+\sin^2\phi_o}\;\Theta_{o2}(\theta_o)\,\Phi'_{o2}(\phi_o)}{\sqrt{\sin^2\theta_o+\sin^2\phi_o}} \qquad (6.33)$$

This term is zero for $\phi_o = \pi/2$, $3\pi/2$. Thus equations (6.19) and (6.21) do not give the dominant behavior of the $\overline{H}$ field and current when the source is in the z = 0 plane and in the $\hat{\theta}_o$ direction. Since the source term is also zero for a source in the $\hat{r}_o$ direction, it is zero for any source in the z = 0 plane and oriented parallel to it. This is reasonable when one considers the symmetrical nature of the source and the antisymmetrical nature of the current shown in Figure 12. The dominant current behavior for this type of source corresponds to the eigenvalue $\nu = 0.296$. The current is

$$\overline{J}(\overline{R}) = \frac{2\kappa Y_o\,(0.172)\,(\overline{N}_{e1}^{II}(\overline{R}_o)\cdot\hat{a})(\kappa r)^{0.296}\,\Phi_{e1}(\phi)\,\hat{r}}{\sin\phi} \qquad \kappa r << 1 \qquad (6.34)$$

The corresponding $\overline{H}$ field is

$$\overline{H}(\overline{R}) = -\kappa\,Y_o\,(0.264)(\overline{N}_{e1}^{II}(\overline{R}_o)\cdot\hat{a})(\kappa r)^{0.296}\,\overline{m}_{e1} \qquad \kappa r << 1 \qquad (6.35)$$

These terms are not singular at the origin. The edge behavior of the $\overline{H}$ field is the same as before, however, and is consistent with half-plane theory. The surface current is in the $\hat{r}$ direction only, and goes to zero at the tip. Its behavior for $\kappa r << 1$ is sketched in Figure 13, and the current flow is sketched in Figure 14.

The dominant fields and currents near the tip of the quarter plane for the various source locations are summarized on pages 118, 119, and 120.

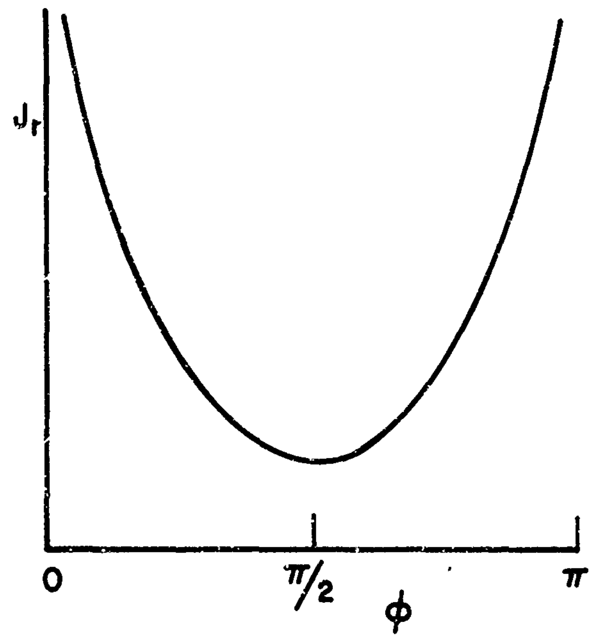

Fig. 13. Variation of J_r for $\kappa r \ll 1$ (From equation 6.34)

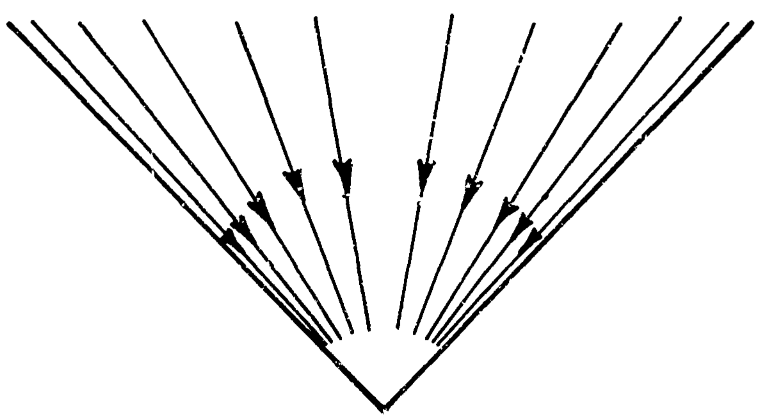

Fig. 14. Surface current flow near the tip of the quarter plane (From equation 6.34)

Summary: Example 1

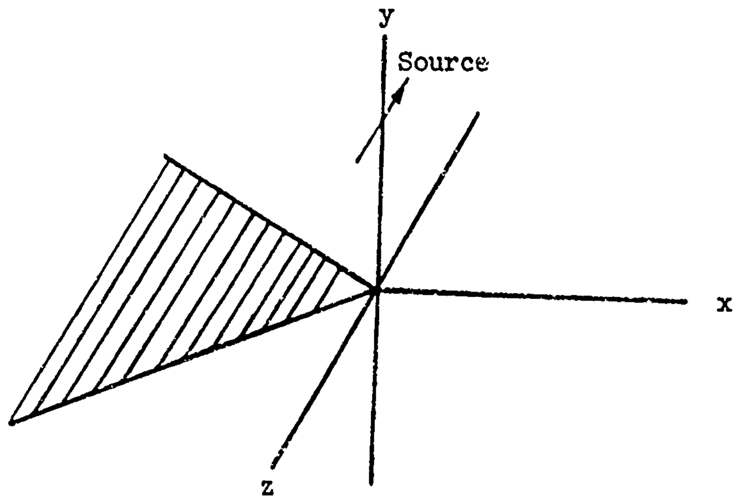

Fig. 15. Source in the z=0 plane and perpendicular to that plane

$$\overline{E}(\overline{R})=j\kappa(0.011)\,(\overline{N}^{II}_{e1}(\overline{R}_o)\cdot\hat{a})(\kappa r)^{0.13}\,[\overline{\ell}_{e1}+2.13\,(\hat{r}\times\overline{m}_{e1})]$$

$$\nu = 1.13\,,\quad \kappa r << 1 \qquad (6.29)$$

$$\overline{H}(\overline{R}) = -\kappa\, Y_o\,(0.024)(\overline{M}^{II}_{o2}(\overline{R}_o)\cdot\hat{a})(\kappa r)^{-0.186}$$

$$[\,\overline{\ell}_{o2} + 1.814\,(\hat{r}\times\overline{m}_{o2})\,]\quad \nu = 0.814\,,\ \kappa r << 1 \qquad (6.19)$$

$$\overline{J}(R) = -2\,\kappa\, Y_o\,(0.031)(\overline{M}^{II}_{o2}(\overline{R}_o)\cdot\hat{a})(\kappa r)^{-0.186}$$

$$\left[-\Phi_{o2}(\phi)\,\hat{\phi} + 1.228\,\frac{\sqrt{1+\sin^2\phi}}{\sin\phi}\,\Phi'_{o2}(\phi)\,\hat{r}\right] \qquad (6.21)$$

$$\nu = 0.814\,,\quad \kappa r << 1$$

Summary: Example 2

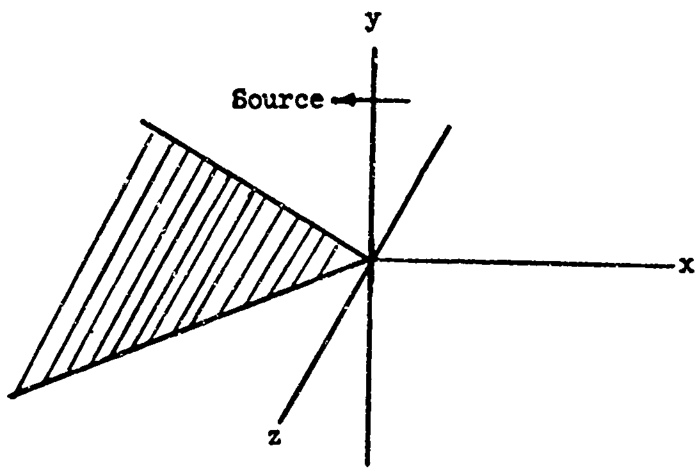

Fig. 16. Source in the z=0 plane and parallel to that plane

$$\bar{E}(\bar{R}) = j\,\kappa\,(0.264)(\bar{N}_{e1}^{II}(\bar{R}_o)\cdot\hat{a})(\kappa r)^{-0.704}$$

$$[\,\bar{\ell}_{e1} + 1.296\,(\hat{r}\times\bar{m}_{e1})\,] \quad \nu = 0.296\ ,\ \kappa r << 1 \qquad (6.18)$$

$$\bar{H}(\bar{R}) = -\kappa\,Y_o\,(0.264)(\bar{N}_{e1}^{II}(\bar{R}_o)\cdot\hat{a})(\kappa r)^{0.296}\,\bar{m}_{e1}$$

$$\nu = 0.296\ ,\quad \kappa r << 1 \qquad (6.35)$$

$$\bar{J}(\bar{R}) = \frac{2\kappa Y_o\,(0.172)\,(\bar{N}_{e1}^{II}(\bar{R}_o)\cdot\hat{a})(\kappa r)^{0.296}\,\Phi_{e1}(\phi)\,\hat{r}}{\sin\phi}$$

$$\nu = 0.296\ ,\quad \kappa r << 1 \qquad (6.34)$$

Summary: Example 3

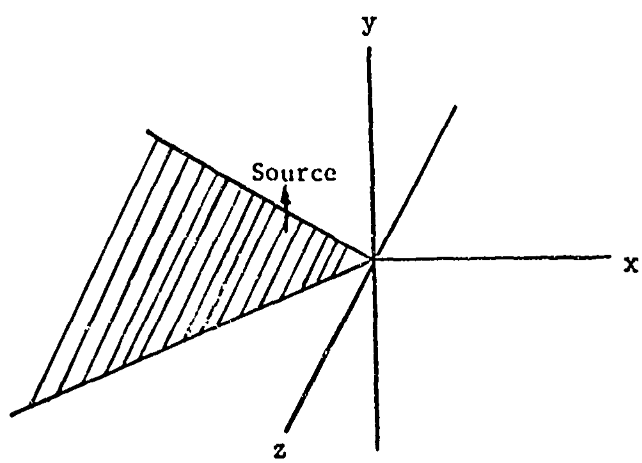

Fig. 17. Source location and orientation arbitrary to the extent that it is not in the z=0 plane

$$\overline{E}(\overline{R}) = j\,\kappa\,(0.264)(\overline{N}_{e1}^{II}(\overline{R}_o)\cdot\hat{a})(\kappa r)^{-0.704}$$

$$\left[\,\overline{\ell}_{e1} + 1.296\,(\hat{r}\times\overline{m}_{e1})\,\right] \quad \nu = 0.296\,,\ \kappa r << 1 \qquad (6.18)$$

$$\overline{H}(\overline{R}) = -\kappa\,Y_o\,(0.024)(\overline{M}_{o2}^{II}(\overline{R}_o)\cdot\hat{a})(\kappa r)^{-0.186}$$

$$\left[\,\overline{\ell}_{o2} + 1.814\,(\hat{r}\times\overline{m}_{o2})\,\right] \quad \nu = 0.814\,,\ \kappa r << 1 \qquad (6.19)$$

$$\overline{J}(R) = -2\,\kappa\,Y_o\,(0.031)(\overline{M}_{o2}^{II}(\overline{R}_o)\cdot\hat{a})(\kappa r)^{-0.186}$$

$$\left[-\Phi_{o2}(\phi)\,\hat{\phi} + 1.228\,\frac{\sqrt{1+\sin^2\phi}}{\sin\phi}\,\Phi'_{o2}(\phi)\,\hat{r}\right] \qquad (6.21)$$

$$\nu = 0.814\,,\ \kappa r << 1$$

Fields and Currents Near the Tip of Angular Sectors with $k^2 = 0.1$ and $k^2 = 0.9$ for a Unit Dipole Source

The dominant behavior of the $\overline{E}$ field in the vicinity of the tip is determined by the lowest even Dirichlet eigenvalue, and the dominant behavior of the $\overline{H}$ field and current is governed by the lowest odd Neumann eigenvalue. In Chapter III, these eigenvalues were determined for $k^2 = 0.1$ and $k^2 = 0.9$.

When the parameter $k^2 = 0.1$, the corner angle of the plane angular sector is 0.795π, thus as k^2 becomes smaller the $\overline{E}$ field behavior should be similar to that for the half plane. For $k^2 = 0.1$, $\nu_{e1} = 0.407$, and it is evident that $(\kappa r)^{\nu_{e1}-1}$, corresponding to equation (6.18), is approaching $(\kappa r)^{-0.5}$ as the plane angular sector approaches a half plane. In order to compare this singular part of the $\overline{E}$ field with half-plane theory, consider the $z = 0$ ($\phi = \pi/2, 3\pi/2$) plane. In this plane the unit vector $\hat{r}$ corresponds to the radial vector $\hat{\rho}$ in cylindrical coordinates. The unit vector $\hat{\phi}$ in the sphero-conal system is the same as $-\hat{z}$ in the cylindrical system, and the sphero-conal unit vector $\hat{\theta}$ approximates the cylindrical unit vector $\hat{\phi}$. (See Figures 18 and 19.) Only the $\hat{r}=\hat{\rho}$ and $\hat{\theta}$ components of equation (6.18) exist in this plane. Thus in the cylindrical coordinate system, the singular fields are the $\hat{\rho}$ and $\hat{\phi}$ components. They vary as $(\kappa\rho)^{-1/2}$, which agrees with half-plane theory. For a general discussion of field behavior at an edge, see Jones[19].

The dominant current behaves like equation (6.21). The eigenvalue corresponding to this equation for $k^2 = 0.1$ is $\nu_{o2} = 0.613$. This eigenvalue is also approaching 0.5 as k^2 approaches zero. Thus the r dependence of the dominant current is also approaching $(\kappa r)^{-1/2}$.

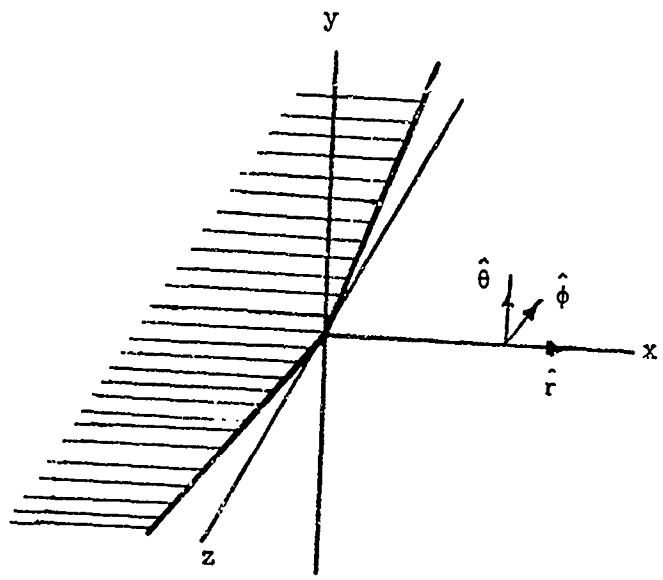

Fig. 18. A plane angular sector with a large corner angle and the unit vectors in the z=0 plane

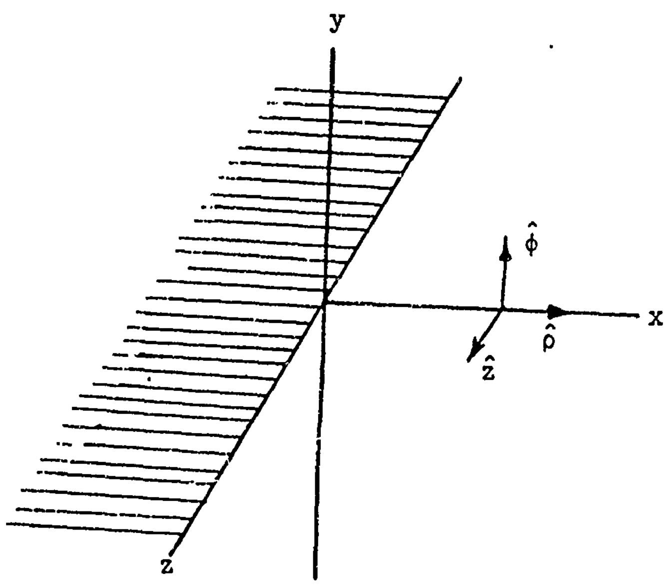

Fig. 19. A half plane in a cylindrical coordinate system and the unit vectors in the z=0 plane

A sketch of this current is given in Figure 20. The current is predominantly parallel to the edge and varying approximately as $(\kappa\rho)^{-1/2}$, which is in agreement with half-plane theory.

The above discussion is rather qualitative and by no means describes the exact behavior of the fields and current as the plane angular sector becomes a half plane. The purpose of the discussion is only to indicate that as $k^2 \to 0$, the various field and current components do appear to be approaching the correct half-plane theory, and that the half-plane problem could be studied if desired.

The lowest even Dirichlet eigenvalue and the lowest odd Neumann eigenvalue were also determined for $k^2 = 0.9$. This corresponds to a plane angular sector with a corner angle of 0.205π. This case will not be considered in detail but will be used only to indicate the variation with r of the fields and currents near the tip of a sharp sector. The eigenvalue $\nu_{e1} = 0.171$, so that the dominant $\overline{E}$ field, which varies as $(\kappa r)^{\nu e1-1}$, is approaching $(\kappa r)^{-1}$. The eigenvalue $\nu_{o2} = 0.970$, so that the dominant $\overline{H}$ and $\overline{J}$, which vary as $(\kappa r)^{\nu o2-1}$, are approaching $(\kappa r)^0$. These are only limiting values, because the vector wave functions corresponding to these eigenvalues go to zero for $k^2 = 1$. The reason for this is that the scattering body is becoming smaller as k^2 approaches 1, and for $k^2 = 1$ there is no scattering body. The singular terms do not exist for this case.

The behavior of the dominant fields and currents is tabulated in Table 7 for $k^2 = 0.1$, 0.5, and 0.9.

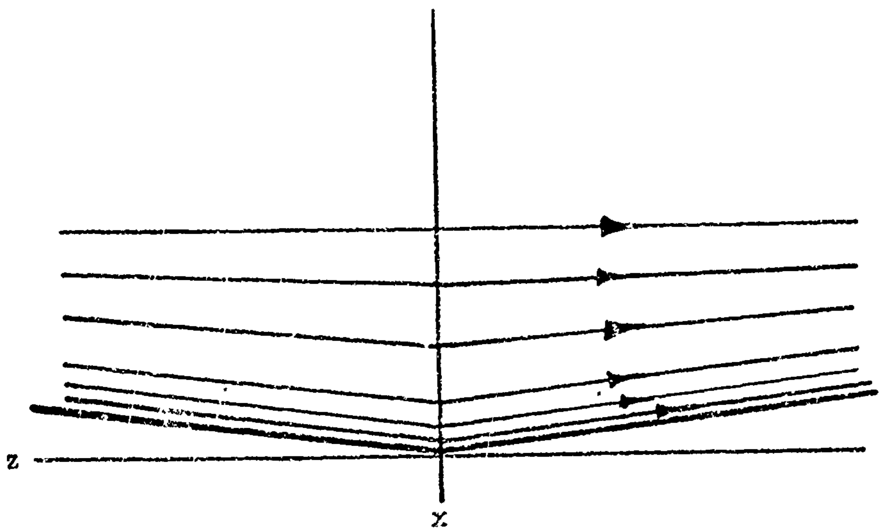

Fig. 20. Surface current flow near the tip of a plane angular sector with a large corner angle (From equation 6.21)

Corner Angle	0.205π	0.500π	0.795π
k^2	0.9	0.5	0.1
ν_{e1}	0.171	0.296	0.407
ν_{o2}	0.970	0.814	0.613
$\overline{E}$	$(\kappa r)^{-0.829}$	$(\kappa r)^{-0.704}$	$(\kappa r)^{-0.593}$
$\overline{H},\overline{J}$	$(\kappa r)^{-0.030}$	$(\kappa r)^{-0.186}$	$(\kappa r)^{-0.387}$

Table 7 - Behavior of Dominant Terms of $\overline{E}$, $\overline{H}$, and $\overline{J}$ for Different Corner Angles

The $\overline{E}$ Field Near the Tip of the Quarter Plane for a Plane Wave Incident

In order to compare the present solution with earlier work, the illumination of the quarter plane by a plane wave will be treated. Consider what happens as the source is moved far from the tip, i.e., as $\kappa r_o \to \infty$. For this case, the large argument approximations can be used for the Hankel functions. In example 1 on page 118, the source term for the $\overline{E}$ field is

$$\overline{N}^{II}_{e1}(\overline{R}_o) \cdot \hat{\phi}_o = 1.414 \frac{(r_o\, h^{(2)}_{1.13}(\kappa r_o))'}{\kappa r_o} \frac{\Theta_{e1}(\theta_o)\, \Phi'_{e1}(\phi_o)}{\sqrt{1 + \sin^2\theta_o}} \tag{6.36}$$

Correct to order $1/\kappa r_o$, the r_o dependence is

$$\frac{(r_o\, h^{(2)}_{1.13}(\kappa r_o))'}{\kappa r_o} \sim \frac{e^{j0.565\pi}\, e^{-j\kappa r_o}}{\kappa r_o} \tag{6.37}$$

and the $\overline{E}$ field due to this source can be written

$$\overline{E}(\overline{R}) = j(0.107)e^{j0.565\pi}\, E^i \frac{\Theta_{e1}(\theta_o)\, \Phi'_{e1}(\phi_o)(\kappa r)^{0.13}}{\sqrt{1 + \sin^2\theta_o}}$$

$$[\ \overline{\ell}_{e1} + 2.13\ (\hat{r} \times \overline{m}_{e1})\ ]\quad \phi_o = \pi/2,\ 3\pi/2,$$

$$\nu = 1.13,\quad \kappa r << 1,\quad \kappa r_o >> 1 \tag{6.38}$$

where

$$E^i = \frac{e^{-j\kappa r_o}}{4\pi\, r_o} \tag{6.39}$$

is the incident field at the tip. E^i also can be considered as the field of a plane wave at the tip. In particular, consider a plane wave propagating in the $-\hat{y}$ direction with the $\overline{E}$ field in the $\hat{z}$ direction. The total field in the vicinity of the tip is

$$\overline{E}(\overline{R}) = j\,(0.096)\,E^{i}\,e^{j0.565\pi}\,(\kappa r)^{0.13}\,(\overline{\ell}_{e1} + 2.13\,(\hat{r} \times \overline{m}_{e1}))$$

$$\nu = 1.13, \quad \kappa r << 1, \quad \kappa r_o >> 1 \tag{6.40}$$

If the plane wave is propagating in the $-\hat{x}$ direction with the same polarization, the $\overline{E}$ field is

$$\overline{E}(\overline{R}) = j\,(0.078)\,E^{i}\,e^{j0.565\pi}(\kappa r)^{0.13}\,(\overline{\ell}_{e1} + 2.13\,(\hat{r} \times \overline{m}_{e1}))$$

$$\nu = 1.13, \quad \kappa r << 1, \quad \kappa r_o >> 1 \tag{6.41}$$

Next consider example 2. For a plane wave propagating in the $-\hat{y}$ direction with $\hat{x}$ polarization, the total $\overline{E}$ field in the vicinity of the tip is

$$\overline{E}(\overline{R}) = j(1.065)\,E^{i}\,e^{j0.148\pi}(\kappa r)^{-0.704}\,(\overline{\ell}_{e1} + 1.296\,(\hat{r} \times \overline{m}_{e1}))$$

$$\nu = 0.296, \quad \kappa r << 1, \quad \kappa r_o >> 1 \tag{6.42}$$

Equations (6.38)-(6.42) really yield no new information concerning the behavior of the fields very close to the tip. For the dominant field terms, the range of the source for any of the three cases considered on pages 118, 119, and 120 affects only the amplitude and not the form of the fields. However, several authors have studied the behavior of the scalar fields near the tip for scalar plane wave illumination. Braunbeck[5] considered the form of the eigenfunction solution of Laplace's equation. Using physical reasoning, he determined that the solution varies as r^{α} ($\alpha < 0.5$). Radlow[3] used an integral transform technique to solve the scalar problem. He interprets his result as being the scattered $\overline{E}$ field due to an incident plane wave with the $\overline{E}$ field parallel to the quarter plane. He estimates that the $\overline{E}$-field varies roughly as $r^{0.25}$. Radlow also mentions another calculation in which Noble estimates the variation to be $r^{0.3}$. The eigenvalue in the present paper which corresponds to these exponents is $\nu_{e1} = 0.296$. Radlow's treatment of the scalar problem appears to be correct and the results he obtains are a valuable contribution to this problem; however his interpretation of his solution as one component of the electric field is clearly in error. The vector solution given here shows that the scattered field depends on the polarization of the incident plane wave and that it varies as $r^{\nu-1}$.

Far Zone Fields and Currents Due to a Unit Dipole Source Close to the Tip of the Quarter Plane

Next the reciprocal case is studied; i.e., the far zone fields are determined for the source located close to the tip of the quarter plane. Perhaps the simplest way to determine these fields is to start with equation (6.2a). Let r_o be very small and use the small argument approximation for the Bessel functions with argument κr_o. Then let r be very large and use the large argument approximation for the Bessel functions with argument κr. The procedure is essentially the same as before, except that r and r_o are interchanged.

Equation (6.2a) is

$$\overline{E}(\overline{R}) = j\,\kappa \sum_q \left[\overline{M}^{II}_{q2}(\overline{R}) \left(\frac{\overline{M}^{I}_{q2}(\overline{R}_o)\cdot\hat{a}}{\Lambda_{q2}} \right) + \overline{N}^{II}_{q1}(\overline{R}) \left(\frac{\overline{N}^{I}_{q1}(\overline{R}_o)\cdot\hat{a}}{\Lambda_{q1}} \right) \right] \quad r \geq r_o \tag{6.43}$$

The source terms are examined for r_o very small, and it is seen that the dominant term is

$$\overline{N}^{I}_{e1}(\overline{R}_o)\cdot\hat{a} = \frac{j_\nu(\kappa r_o)}{\kappa r_o}\,\overline{\ell}_{e1}\cdot\hat{a} + \frac{(r_o\, j_\nu(\kappa r_o))'}{\kappa r_o}\,(\hat{r}_o \times \overline{m}_{e1})\cdot\hat{a} \tag{6.44}$$

with $\nu = 0.296$. Using the small argument approximation for the Bessel functions, this becomes

$$\overline{N}^{I}_{e1}(\overline{R}_o)\cdot\hat{a} \approx 1.236\,(\kappa r_o)^{-0.704}\left(\overline{\ell}_{e1}\cdot\hat{a} + 1.296\,(\hat{r}_o \times \overline{m}_{e1})\cdot\hat{a}\right) \tag{6.45}$$

The $\overline{R}$ dependent term is

$$\overline{N}^{II}_{e1}(\overline{R}) = \frac{h^{(2)}_\nu(\kappa r)}{\kappa r}\,\overline{\ell}_{e1} + \frac{(r\, h^{(2)}_\nu(\kappa r))'}{\kappa r}\,(\hat{r} \times \overline{m}_{e1}) \tag{6.46}$$

Using the large argument approximations for the Hankel functions and neglecting terms of order higher than $1/\kappa r$, this becomes

$$\overline{N}^{II}_{e1}(\overline{R}) \sim e^{j0.148\pi}\,\frac{e^{-j\kappa r}}{\kappa r}\,(\hat{r} \times \overline{m}_{e1}) \tag{6.47}$$

Now consider the source to be located in the $z = 0$ plane and in the $-\hat{\theta}_o$ direction. This corresponds to example 2 on page 119. In particular, let the source be located on the line $\theta_o = \pi/2$, $\phi_o = \pi/2$, $(-\hat{\theta}_o = \hat{x})$. The $\overline{E}$ field is

$$\overline{E}(\overline{R}) \sim j\, 2.157 \; e^{j0.148\pi} (\kappa r_o)^{-0.704} \; \frac{e^{-j\kappa r}}{4\pi r} \; (\hat{r} \times \overline{m}_{e1}) \tag{6.48}$$

In the $z = 0$ plane this is

$$\overline{E}(\overline{R}) \sim j2.304 \; e^{j0.148\pi} (\kappa r_o)^{-0.704} \; \frac{e^{-j\kappa r}}{4\pi r} \; \Theta'_{e1}(\theta)\, \hat{\theta} \tag{6.49}$$

This pattern is normalized to unity at $\theta = \pi$ and plotted in Figure 21. C. T. Tai[20] has computed the pattern in this plane for the same source but with a half plane scattering body. The two patterns are essentially the same. As a matter of fact, the pattern has the same general shape for any sector angle provided $\kappa r_o << 1$.

In the $x = 0$ plane, the $\overline{E}$ field is

$$\overline{E}(\overline{R}) \sim j\, 1.666 \; e^{j0.148\pi} (\kappa r_o)^{-0.704} \; \frac{e^{-j\kappa r}}{4\pi r}$$

$$\left[-\, \Phi'_{e1}(\phi)\, \hat{\phi} + (0.540)\, \frac{\Phi_{e1}(\phi)\, \hat{\theta}}{\sqrt{1 + \sin^2 \phi}} \right] \tag{6.50}$$

Both components of this pattern are plotted separately in Figures 22 and 23. These patterns are normalized relative to Figure 21. Note that the magnitude of E_ϕ in Figure 22 is quite small, in fact the peak amplitude is more than 20db down from the peak amplitude of the field in the $z = 0$ plane.

The y = 0 plane is composed of four sectors, $\phi = \pi$, $\theta = 0$, $\phi = 0, 2\pi$, and $\theta = \pi$. For $\phi = 0$, the $\overline{E}$ field is

$$\overline{E}(\overline{R}) \sim -j(2.157)\, e^{j0.148\pi} (\kappa r_o)^{-0.704}\, \frac{e^{-j\kappa r}}{4\pi r}$$

$$\frac{\sqrt{1 + \sin^2\theta}}{\sin\theta}\; \Theta'_{e1}(\theta)\, \hat{\theta} \tag{6.51}$$

For $\theta = 0$, the $\overline{E}$ field is

$$\overline{E}(\overline{R}) \sim -j(2.157)\, e^{j0.148\pi} (\kappa r_o)^{-0.704}\, \frac{e^{-j\kappa r}}{4\pi r}$$

$$\frac{\sqrt{1 + \sin^2\phi}}{|\sin\phi|}\; \Phi'_{e1}(\phi)\, \hat{\phi} \tag{6.52}$$

For $\phi = \pi$, the $\overline{E}$ field is given by equation (6.51). The fields for $\phi = 0$, $\theta = 0$, and $\phi = \pi$ are all parallel to the y = 0 plane. For $\theta = \pi$, the parallel field is of course zero. A field perpendicular to the quarter plane does exist, however. It is

$$\overline{E}(\overline{R}) \sim j(1.406)(\kappa r_o)^{-0.704}\, e^{j0.148\pi}\, \frac{e^{-j\kappa r}}{4\pi r}\, \frac{\Phi_{e1}(\phi)\, \hat{\theta}}{|\sin\phi|} \tag{6.53}$$

This field exists on both sides of the quarter plane, the $\overline{E}$ vectors pointing either into or out of the sector on both sides. The patterns given by equations (6.51), (6.52), and (6.53) are normalized relative to Figure 21 and plotted in Figure 24. Note that the edges of the quarter plane appear to have a guiding effect on the field. This might have some useful applications.

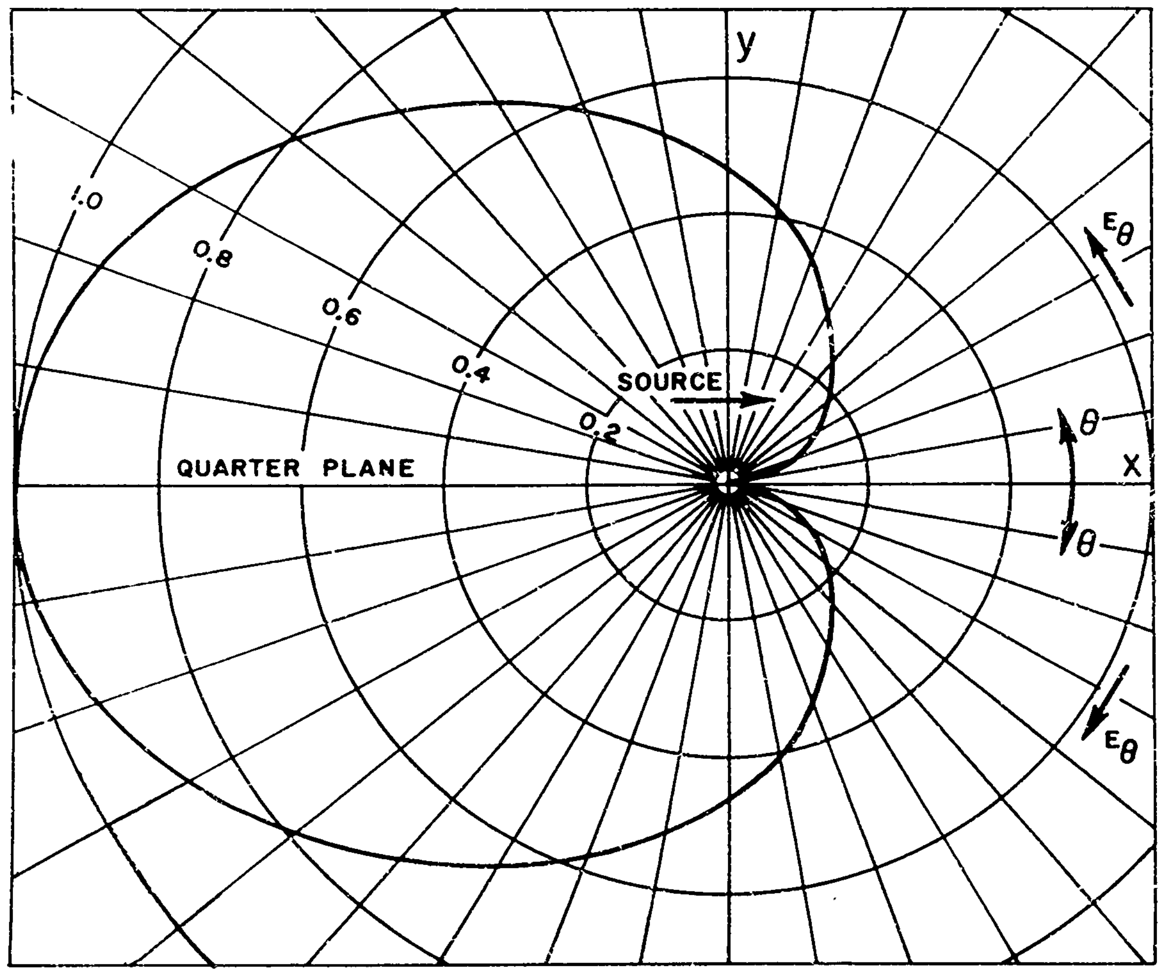

Fig. 21. Far field in the z = 0 plane due to an infinitesimal dipole source close to the tip of the quarter plane

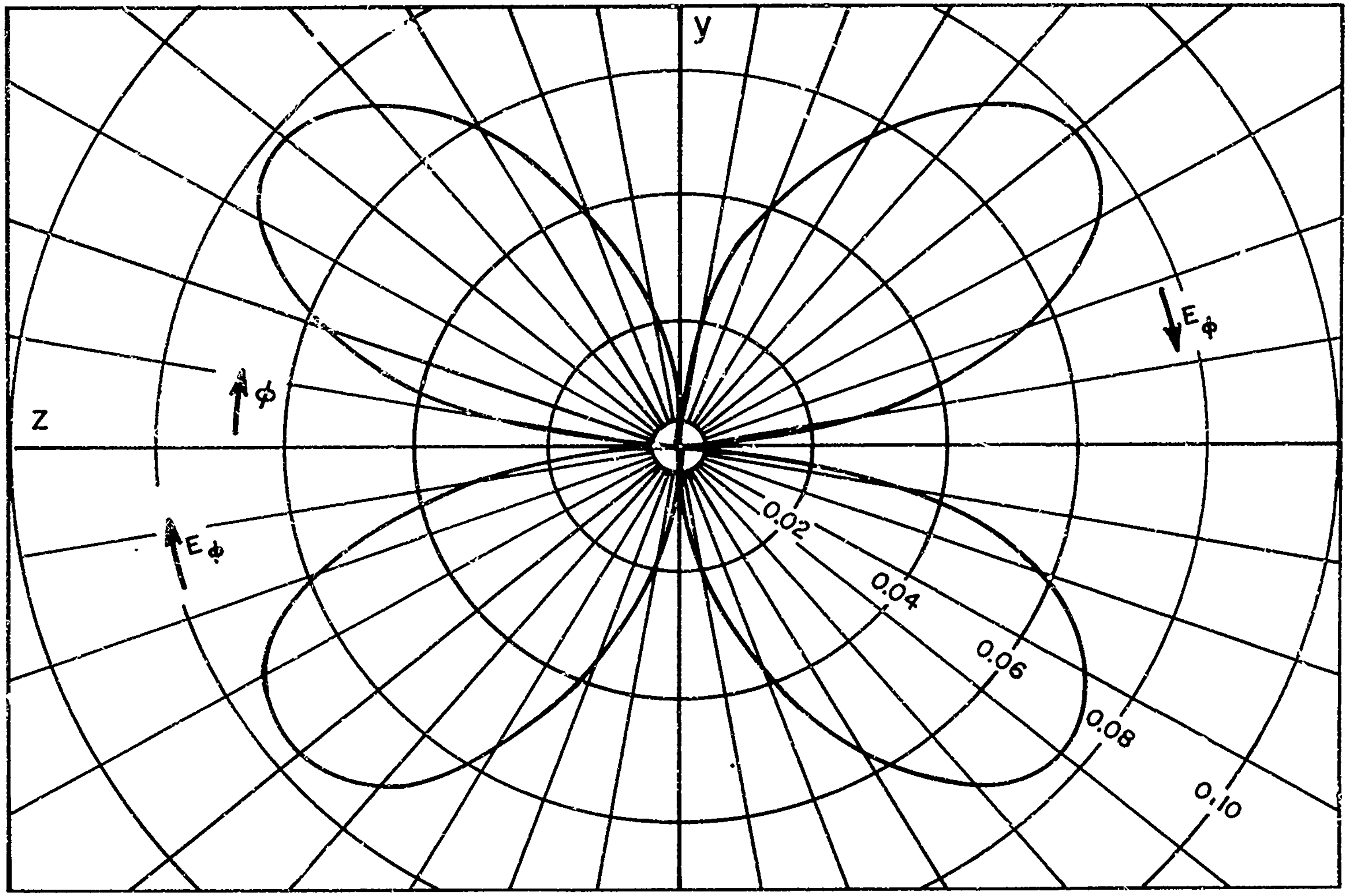

Fig. 22. $\hat{\phi}$ component of the far field in the x = 0 plane due to an infinitesimal dipole source close to the tip of the quarter plane (source in the z = 0 plane and parallel to it)

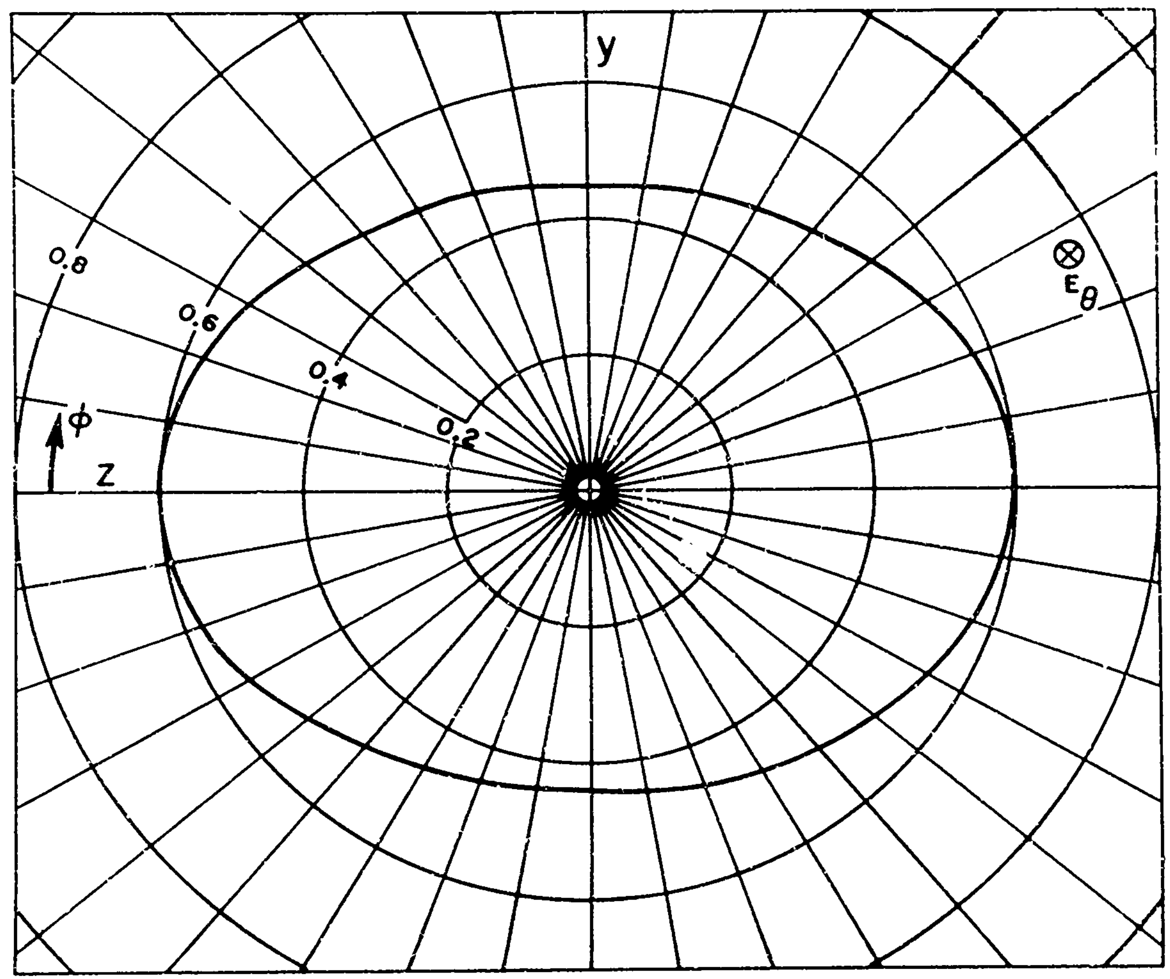

Fig. 23. $\hat{\theta}$ component of the far field in the $x = 0$ plane due to an infinitesimal dipole source close to the tip of the quarter plane (source in the $z = 0$ plane and parallel to it)

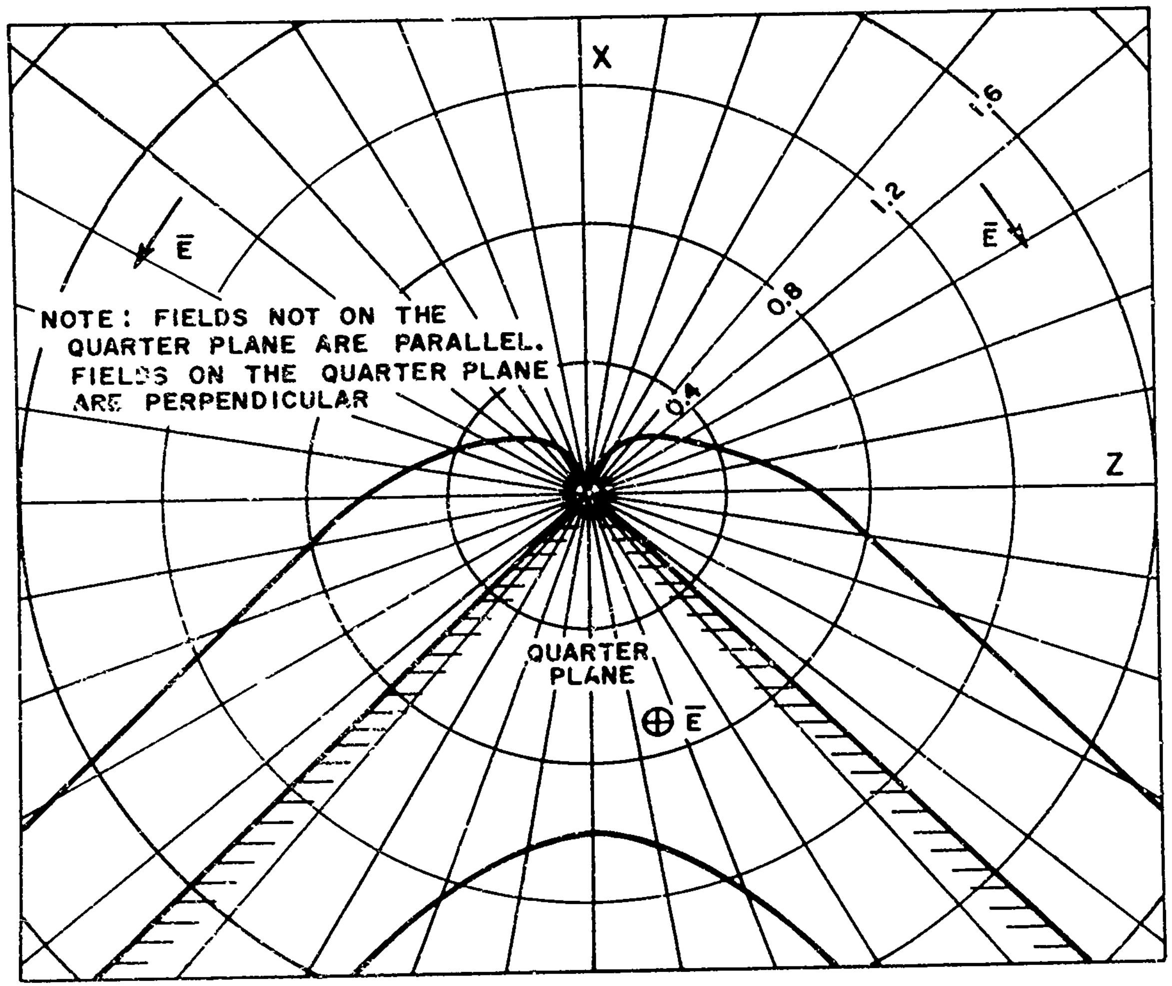

Fig. 24. Far fields in the $y = 0$ plane due to an infinitesimal dipole source close to the tip of the quarter plane (source in the $z = 0$ plane and parallel to it)

The dominant far zone current is

$$\bar{J}(\bar{R}) \sim -2j\, Y_0\, (1.406)\, e^{j0.148\pi} (\kappa r_o)^{-0.704} \frac{e^{-j\kappa r}}{4\pi r}$$

$$\frac{\Phi_{e1}(\phi)}{\sin\phi}\, \hat{r} \tag{6.54}$$

At a constant r, the behavior of $\bar{J}(\bar{R})$ is the same as that shown in Figure 13. Equation (6.34) is valid for the current near the tip due to this source, so Figures 13 and 14 show the behavior of the current there. Notice that the current in the far zone is decaying as 1/r and the current near the tip is varying as $r^{0.296}$; both currents have the same behavior with respect to ϕ.

Next, consider the source location to be on the quarter plane, still in the z = 0 plane and still in the $-\hat{\theta}_o(\hat{y})$ direction. This source corresponds to a vertical dipole on the quarter plane and close to the tip. The only thing that is changed is that $\theta_o = \pi$ instead of $\pi/2$. Equations (6.48)-(6.54) are unchanged except in magnitude. Multiply each of these equations by (2.21) to obtain the far fields and current for this source. Figures 21-24 give the patterns. Actually, these figures give the patterns for any source located in the z = 0 plane and parallel to it, and very close to the tip of the quarter plane. Of course, the amplitude of the field is zero for the obvious case of a source on the quarter plane and parallel to it. The amplitude is also zero for the source on the x axis and oriented in the $\hat{y}$ direction. This is the configuration that produces no scattered field. In Figures 21 to 24 it is interesting to note that the magnitude of $\overline{E}$ increases as the electric current dipole approaches the tip. The dominant term of the electric field for these cases is entirely due to currents induced on the quarter plane. Thus $\overline{E}$ can be interpreted as a field scattered from the quarter plane due to a current dipole in close proximity to the tip.

Now consider the source location shown in example 1 on page 118. Let the source be in the z = 0 plane and perpendicular to it. In particular consider a source in the $-\hat{\phi}_o$ direction and on the $\phi_o = \pi/2$, $\theta_o = \pi/2$ line $(-\hat{\phi}_o = \hat{z})$. The dominant term for the $\overline{E}$ field corresponds to the eigenvalue $\nu_{e1} = 1.13$. The far zone $\overline{E}$ field is

$$\overline{E}(\overline{R}) \sim j(0.199)\, e^{j0.565\pi} (\kappa r_o)^{0.13} \frac{e^{-j\kappa r}}{4\pi r} (\hat{r} \times \overline{m}_{e1}) \tag{6.55}$$

In the z = 0 plane this is

$$\overline{E}(\overline{R}) \sim j(0.291)\, e^{j0.565\pi} (\kappa r_o)^{0.13} \frac{e^{-j\kappa r}}{4\pi r} \frac{\Theta_{e1}(\theta)\, \hat{z}}{\sqrt{1+\sin^2\theta}} \tag{6.56}$$

This pattern is normalized to unity and plotted in Figure 25. The pattern in Figure 25 also agrees with Tai's half plane pattern, which is to be expected, because the general pattern shape is good for all sector angles.

In the x = 0 plane, the far zone field is

$$\overline{E}(\overline{R}) \sim j(0.243)\, e^{j0.565\pi} (\kappa r_o)^{0.13} \frac{e^{-j\kappa r}}{4\pi r}$$

$$\left[\Phi_{e1}'(\phi)\, \hat{\phi} - \frac{0.332\, \Phi_{e1}(\phi)\, \hat{\theta}}{\sqrt{1 + \sin^2\phi}} \right] \tag{6.57}$$

Both components of this pattern are normalized relative to Figure 25 and plotted separately in Figures 26 and 27. The quarter plane appears to have very little effect on Figure 26. The pattern in Figure 27 is of a cross-polarized field and is down about 10db from the normally polarized field.

In the y = 0 plane, the far field for the $\phi = 0$ sector is

$$\overline{E}(\overline{R}) \sim j(0.199)\, e^{j0.565\pi}(\kappa r_o)^{0.13}\, \frac{e^{-j\kappa r}}{4\pi r}\, \frac{\sqrt{1+\sin^2\theta}}{\sin\theta}\, \Theta'_{e1}(\theta)\, \hat{\theta} \tag{6.58}$$

For the $\theta = 0$ sector, it is

$$\overline{E}(\overline{R}) \sim j(0.199)\, e^{j0.565\pi}(\kappa r_o)^{0.13}\, \frac{e^{-j\kappa r}}{4\pi r}\, \frac{\sqrt{1+\sin^2\phi}}{|\sin\phi|}\, \Phi'_{e1}(\phi)\, \hat{\phi} \tag{6.59}$$

For $\phi = \pi$, the field is given by equation (6.58). These fields behave in the same way as in the previous example, i.e., they are all parallel to the y = 0 plane. For $\theta = \pi$, there is a perpendicular field given by

$$\overline{E}(\overline{R}) \sim -j(0.195)\, e^{j0.565\pi}(\kappa r_c)^{0.13}\, \frac{e^{-j\kappa r}}{4\pi r}\, \frac{\Phi_{e1}(\phi)\, \hat{\theta}}{|\sin\phi|} \tag{6.60}$$

Again, this field exists on both sides of the quarter plane, the $\overline{E}$ vectors pointing either into or out of the sector on both sides.

The patterns given by equations (6.58), (6.59), and (6.60) are normalized relative to Figure 25 and plotted in Figure 28. Again note the guiding effect of the edges of the quarter plane.

The current is

$$\overline{J}(\overline{R}) \sim 2\, j\, Y_o (0.195)\, e^{j0.565\pi} (\kappa r_c)^{0.13} \, \frac{e^{-j\kappa r}}{4\pi r}$$

$$\frac{\Phi_{e1}(\phi)\, \hat{r}}{\sin\phi} \tag{6.61}$$

At a constant r, the behavior of $\overline{J}(\overline{R})$ is similar to that shown in Figure 10. Near the tip, equation (6.24) is valid for this source, so Figures 10, 11, and 12 show the behavior of the current there. The current in the far zone is decaying as $1/r$; the current near the tip is singular, varying as $(\kappa r)^{-0.186}$.

One other source location is considered corresponding to this case. Let this source be located on the x axis and oriented in the $\hat{z}$ direction. Equations (6.55)-(6.61) have the same form, only their magnitude being changed. Multiply these equations by (0.819) to obtain the far fields and currents for this source. Figures 25-28 give the patterns for this source and, in fact, for any source in the $z = 0$ plane and perpendicular to it, and very close to the tip of the quarter plane. Again it is mentioned that the amplitude of the field is zero for the obvious case of the source on the quarter plane. In Figures 25-28 it is interesting to note that the magnitude of $\overline{E}$ decreases as the electric current dipole approaches the tip. The dominant term of the electric field for these cases again is entirely

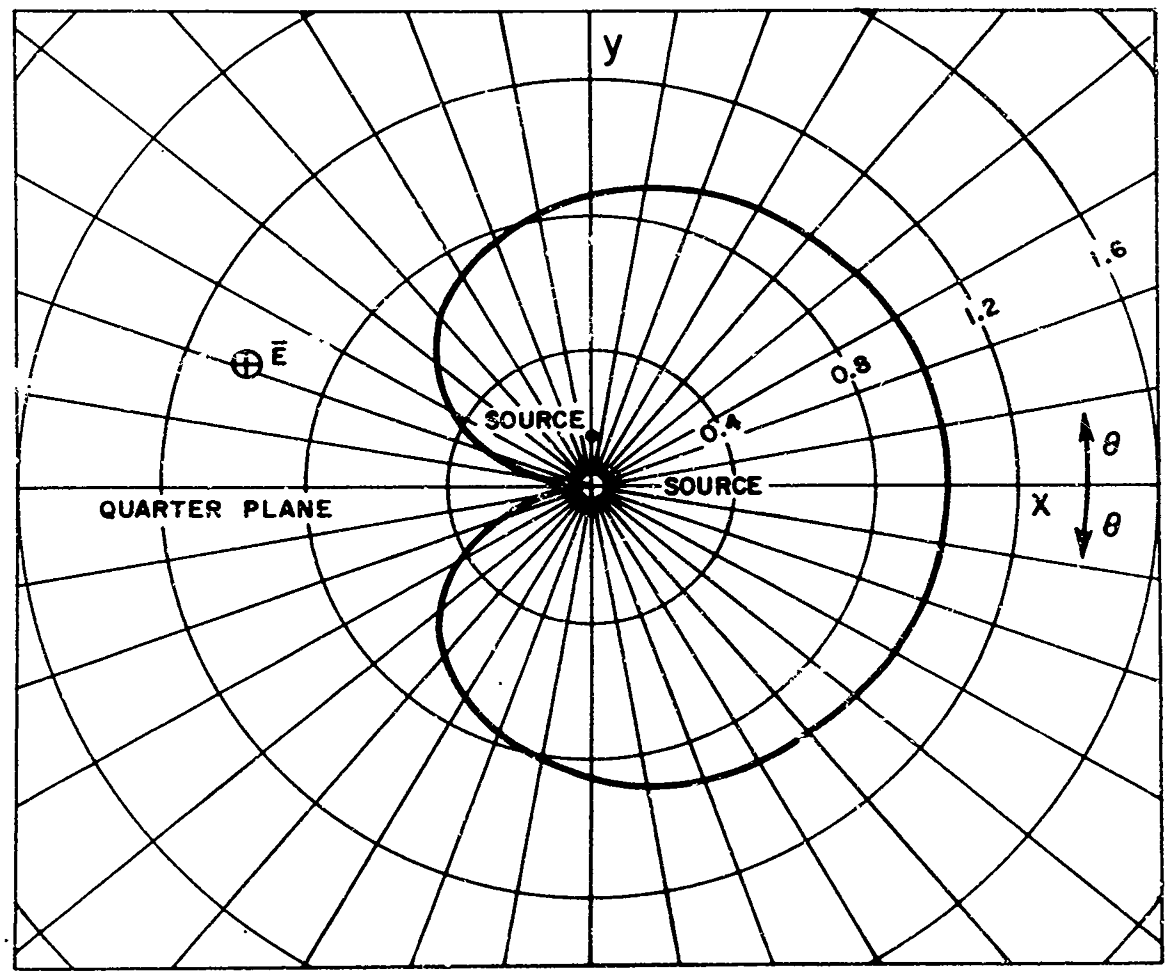

Fig. 25. Far field in the $z = 0$ plane due to an infinitesimal dipole source close to the tip of the quarter plane (source in the $z = 0$ plane and perpendicular to it)

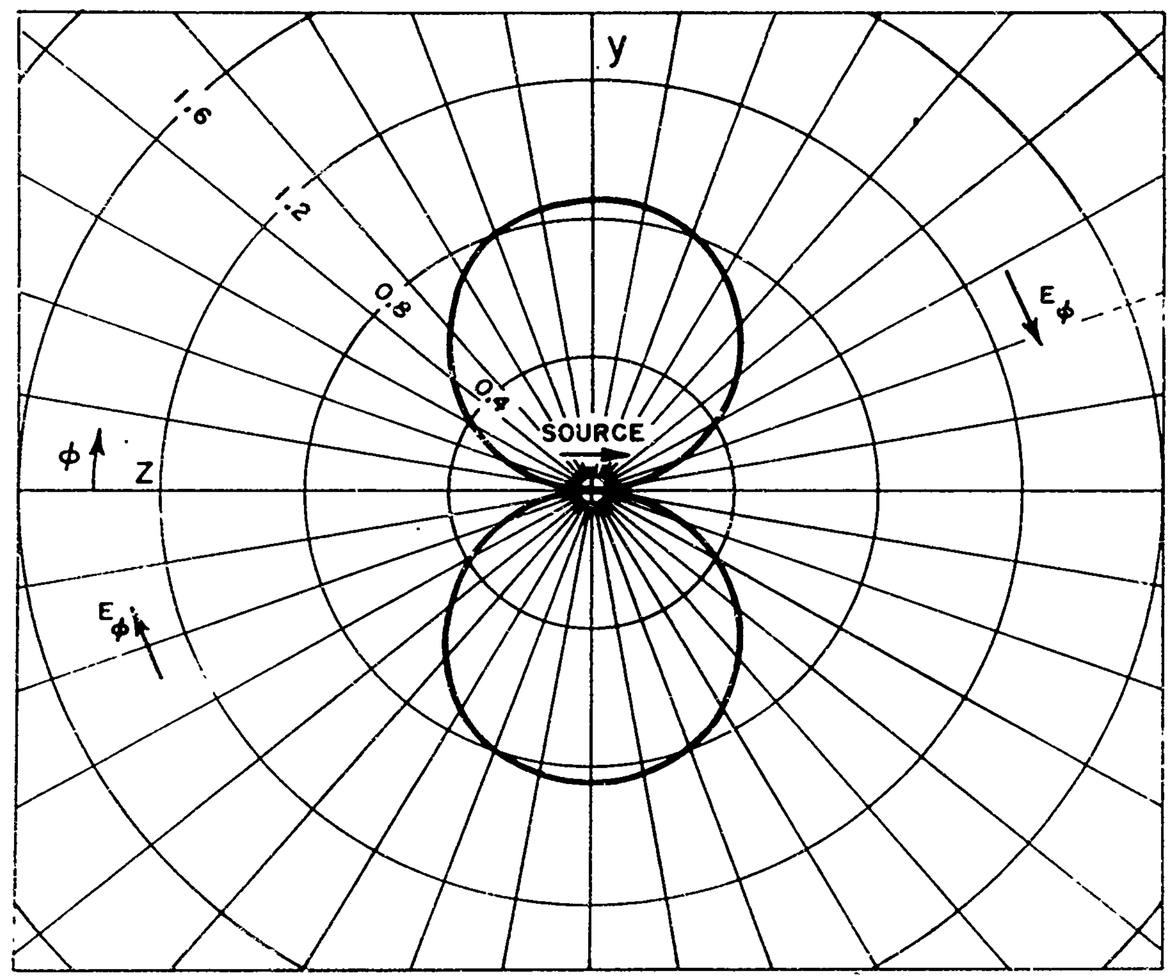

Fig. 26. $\hat{\phi}$ component of the far field in the x = 0 plane due to an infinitesimal dipole source close to the tip of the quarter plane (source in the z = 0 plane and perpendicular to it)

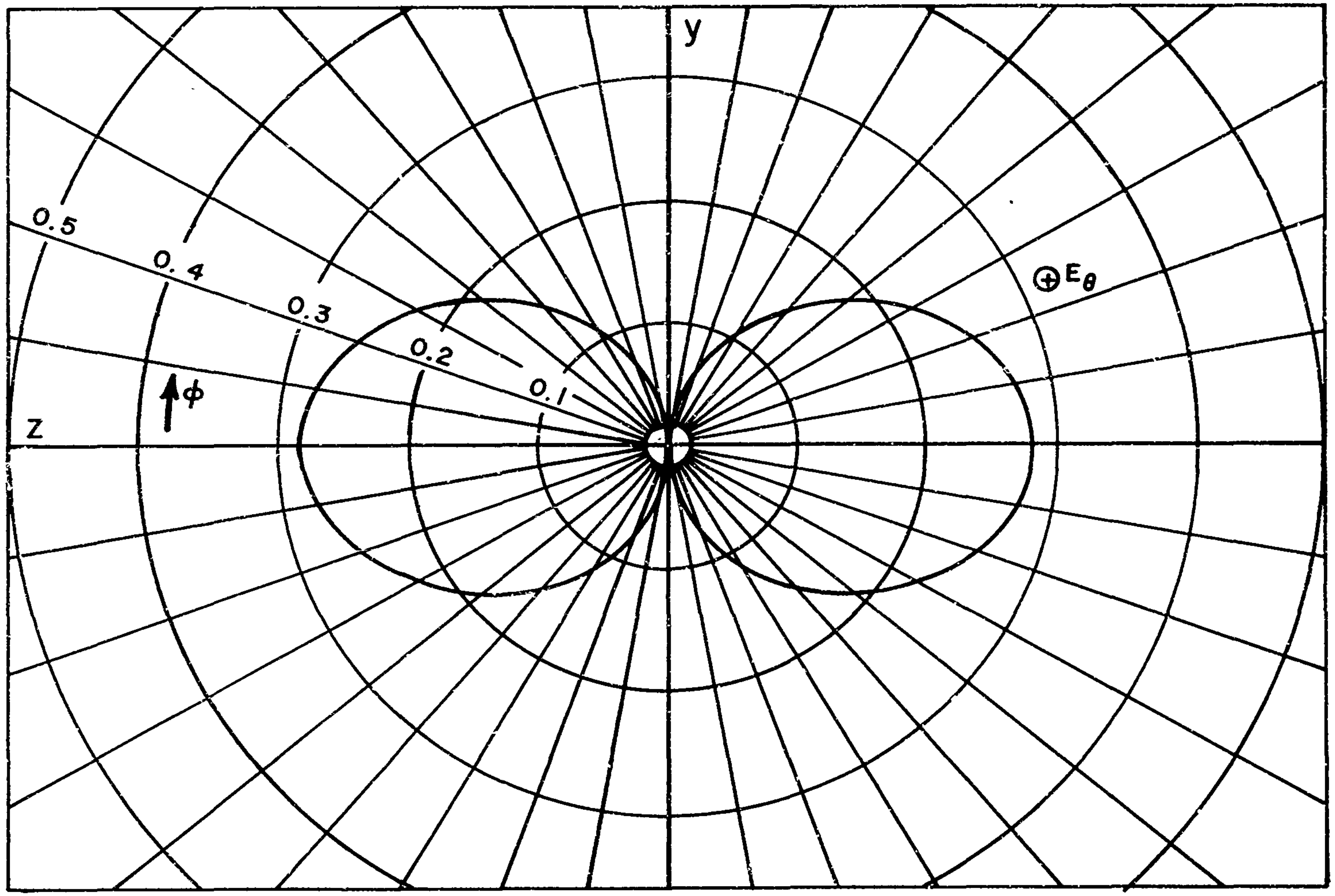

Fig. 27. $\hat{\theta}$ component of the far field in the x = 0 plane due to an infinitesimal dipole source close to the tip of the quarter plane (source in the z = 0 plane and perpendicular to it)

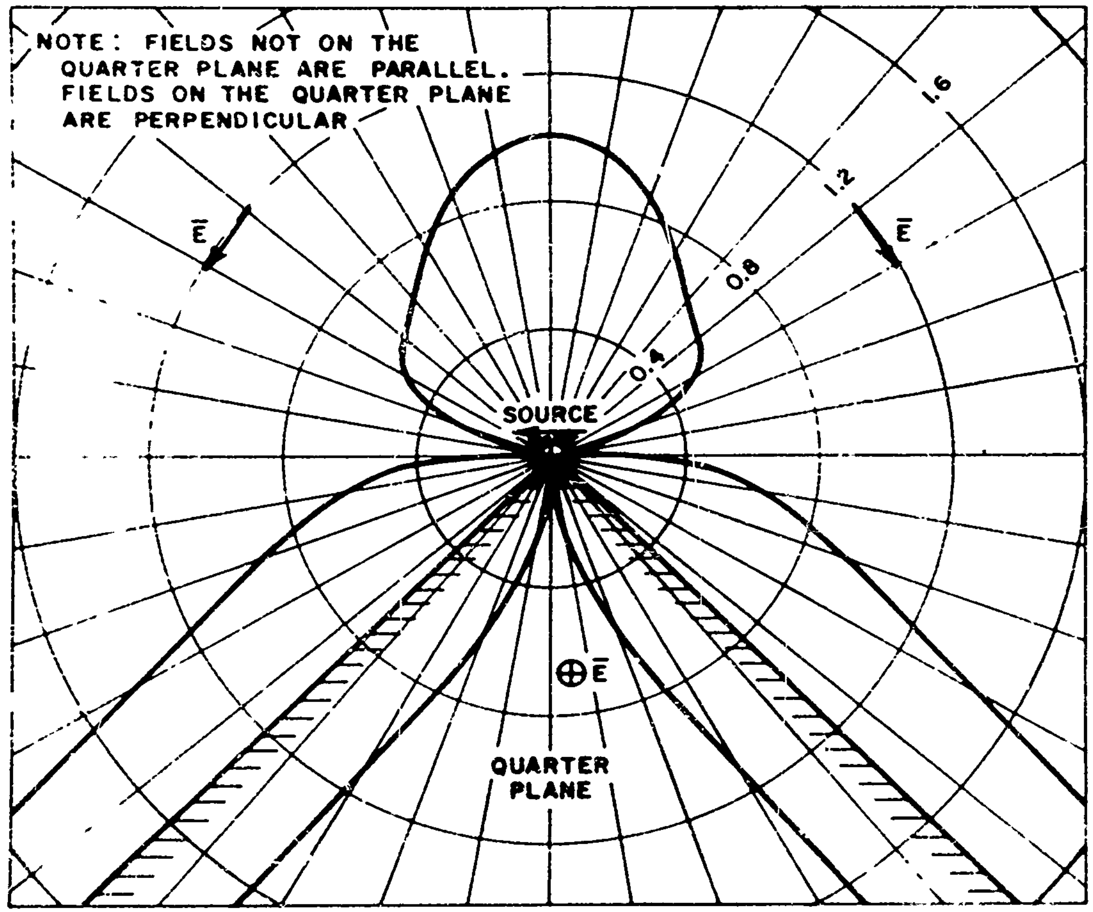

Fig. 28. Far fields in the y = 0 plane due to an infinitesimal dipole source close to the tip of the quarter plane (source in the z = 0 plane and perpindicular to it)

due to currents induced on the quarter plane, and $\overline{E}$ can be interpreted as a scattered field for these cases too.

CHAPTER VII

CONCLUDING REMARKS

In this report a rigorous, general solution for the diffraction of an electromagnetic wave by a perfectly-conducting plane angular sector is derived in the form of a dyadic Green's function. Thus, for example, the excitation of the plane angular sector by neighboring antennas or by slots in the sector can be determined. This problem has not been solved previously and although formal solutions to the corresponding scalar problem exist, no numerical results have been presented.

The dyadic Green's function is expressed in terms of a complete set of vector wave functions of the sphero-conal coordinate system. The vector wave functions contain Lamé functions, which have been studied numerically in the case of the quarter plane; the results are presented in tables of the Lamé functions and their associated eigenvalues. Except for the Lamé polynomials, the Lamé functions and their eigenvalues have not been tabulated previously.

Numerical results for the behavior of the fields and currents near the tip of the quarter plane are obtained and compared with estimates and conjectures made by other authors.

The patterns of electric current dipoles very close to the tip are presented. In the perpendicular plane of symmetry, the pattern is the same as that of the half plane similarly excited. The patterns in the vicinity of the edges of the quarter plane indicate that a substantial amount of energy is guided away from the source along these edges. In general, a dipole very close to the quarter plane tip excites a strong surface current flow along the edges and near the tip; consequently, the pattern of the electric current dipole is severely altered by the presence of the quarter plane.

The examples that have been given are actually quite simple. They are restricted such that either the field point or the source point is very close to the tip of the quarter plane; in fact, so close that only the first or second term in the expansion is used. From the tables in Chapter III it is seen that approximately 200 terms of the vector wave function expansion have been computed. Using these results it should be possible to calculate the fields of sources up to one wavelength from the tip with good accuracy. At this distance it may be possible to determine the diffraction coefficient approximately and thereby extend the solution by means of the geometrical theory of diffraction so that the fields of sources remote from the tip can be calculated. All of this, however, must await future work.

APPENDIX A

THE SELF-ADJOINT AND POSITIVE DEFINITE PROPERTY OF THE TWO-DIMENSIONAL STURM-LIOUVILLE OPERATOR

The differential equation of interest can be written

$$L\, y - \lambda\, \rho\, y = 0 \tag{A1}$$

where the Sturm-Liouville operator is

$$L = \frac{-1}{\sqrt{1 - k'^2 \cos^2 \phi}} \frac{\partial}{\partial \theta} \left(\sqrt{1 - k^2 \cos^2 \theta}\, \frac{\partial}{\partial \theta} \right) \tag{A2}$$

$$- \frac{1}{\sqrt{1 - k^2 \cos^2 \theta}} \frac{\partial}{\partial \phi} \left(\sqrt{1 - k'^2 \cos^2 \phi}\, \frac{\partial}{\partial \phi} \right)$$

and the weight function is

$$\rho = \frac{k^2 \sin^2 \theta + k'^2 \sin^2 \phi}{\sqrt{1 - k^2 \cos^2 \theta}\, \sqrt{1 - k'^2 \cos^2 \phi}} \tag{A3}$$

Let y be an eigenfunction

$$y = \Theta(\theta)\ \ \Phi(\phi) \tag{A4}$$

corresponding to an eigenvalue

$$\lambda = \nu(\nu+1) \tag{A5}$$

Equation (A1) is the same as equation (3.5). The weight function is the same ρ defined by equation (5.29).

The self-adjoint property is defined by

$$< w, L\, y > - < y, L\, w > = 0 \tag{A6}$$

where $< >$ denotes a scalar product, and y and w are eigenfunctions.

$$< w, L\, y > = - \int_{-\pi}^{\pi} \int_{0}^{\pi} \left[\frac{w}{\sqrt{1 - k'^2 \cos^2 \phi}} \frac{\partial}{\partial \theta} \left(\sqrt{1 - k^2 \cos^2 \theta}\, \frac{\partial y}{\partial \theta} \right) \right.$$

$$\left. + \frac{w}{\sqrt{1 - k^2 \cos^2 \theta}} \frac{\partial}{\partial \phi} \left(\sqrt{1 - k'^2 \cos^2 \phi}\, \frac{\partial y}{\partial \phi} \right) \right] d\theta\, d\phi \tag{A7}$$

Performing the differentation and rearranging the integrals, this can be written

$$< w, L\, y > = - \int_{-\pi}^{\pi} \frac{1}{\sqrt{1 - k'^2 \cos^2 \phi}} \int_{0}^{\pi} \left[\sqrt{1 - k^2 \cos^2 \theta}\; w \frac{\partial^2 y}{\partial \theta^2} \right.$$

$$\left. + \frac{k^2 \cos \theta \sin \theta}{\sqrt{1 - k^2 \cos^2 \theta}}\; w \frac{\partial y}{\partial \theta} \right] d\theta\, d\phi - \int_{0}^{\pi} \frac{1}{\sqrt{1 - k^2 \cos^2 \theta}}$$

$$\int_{-\pi}^{\pi} \left[\sqrt{1 - k'^2 \cos^2 \phi}\; w \frac{\partial^2 y}{\partial \phi^2} + \frac{k'^2 \cos \phi \sin \phi}{\sqrt{1 - k'^2 \cos^2 \phi}}\; w \frac{\partial y}{\partial \phi} \right] d\phi\, d\theta \tag{A8}$$

Integrating the first term of the first θ integral and the first term of the second ϕ integral by parts, this becomes

$$< w, L\ y > = -\int_{-\pi}^{\pi} \frac{1}{\sqrt{1 - k'^2 \cos^2 \phi}} \left[\sqrt{1 - k^2 \cos^2 \theta}\ w \frac{\partial y}{\partial \theta} \Big|_{\theta=0}^{\theta=\pi} \right.$$

$$\left. - \int_0^{\pi} \sqrt{1 - k^2 \cos^2 \theta}\ (\frac{\partial w}{\partial \theta})(\frac{\partial y}{\partial \theta})\ d\theta \right] d\phi \ - \int_0^{\pi} \frac{1}{\sqrt{1 - k^2 \cos^2 \theta}}$$

$$\left[\sqrt{1 - k'^2 \cos^2 \phi}\ w \frac{\partial y}{\partial \phi} \Big|_{\phi=-\pi}^{\phi=\pi} - \int_{-\pi}^{\pi} \sqrt{1 - k'^2 \cos^2 \phi}\ (\frac{\partial w}{\partial \phi})(\frac{\partial y}{\partial \phi})\ d\phi \right] d\theta \tag{A9}$$

For both the Dirichlet and the Neumann problems, there are four possible combinations of w and y. Either they are both even, both odd, w even and y odd, or w odd and y even. For all of these combinations for both problems, the endpoint contributions are zero, or the entire integral is zero. In any case, $< w, L\ y >$ can be written

$$< w, L\ y > = \int_{-\pi}^{\pi} \int_0^{\pi} \left[\frac{\sqrt{1 - k^2 \cos^2 \theta}}{\sqrt{1 - k'^2 \cos^2 \phi}} (\frac{\partial w}{\partial \theta})(\frac{\partial y}{\partial \theta}) \right.$$

$$\left. + \frac{\sqrt{1 - k'^2 \cos^2 \phi}}{\sqrt{1 - k^2 \cos^2 \theta}} (\frac{\partial w}{\partial \phi})(\frac{\partial y}{\partial \phi}) \right] d\theta\ d\phi \tag{A10}$$

It is easily seen from the symmetrical nature of the equation with respect to w and y, that it is also equal to $< y, L\ w >$, and therefore

$$< w, L\ y > = < y, L\ w > \tag{A11}$$

and L is a self-adjoint operator.

In order to determine the positive-definite property of L, consider equation (A1). Scalar multiplication by y yields

$$\frac{< y, L y >}{< y, \rho y >} = \lambda \tag{A12}$$

where y is the eigenfunction corresponding to the eigenvalue λ.

$$< y, \rho y > = \int_{-\pi}^{\pi} \int_{0}^{\pi} \frac{y^2 (k^2 \sin^2 \theta + k'^2 \sin^2 \phi)}{\sqrt{1 - k^2 \cos^2 \theta} \sqrt{1 - k'^2 \cos^2 \phi}} d\theta \, d\phi \tag{A13}$$

$$< y, L y > = \int_{-\pi}^{\pi} \int_{0}^{\pi} \left[\frac{\sqrt{1 - k^2 \cos^2 \theta}}{\sqrt{1 - k'^2 \cos^2 \phi}} \left(\frac{\partial y}{\partial \theta}\right)^2 \right.$$

$$\left. + \frac{\sqrt{1 - k'^2 \cos^2 \phi}}{\sqrt{1 - k^2 \cos^2 \theta}} \left(\frac{\partial y}{\partial \phi}\right)^2 \right] d\theta \, d\phi \tag{A14}$$

As a point of interest note that

$$< y, L y > = \Lambda \tag{A15}$$

where Λ is the normalization constant employed in the dyadic Green's function.

Inspection of equations (A13) and (A14) shows that the integrands are always positive and consequently

$$\frac{< y, L y >}{< y, \rho y >} = \lambda \geq 0 \tag{A16}$$

Thus the operator L is positive definite and λ is always greater than zero. Recall that $\lambda = \nu(\nu+1)$ so that equation (A16) implies that

$$\nu \geq 0 \tag{A17}$$

or

$$\nu \leq -1$$

Note that equation (A1) is symmetric around $\nu = -1/2$, that is, the differential equation is the same if ν is replaced by $-(\nu+1)$. In other words, for $\nu = -0.4$ or -0.6, or for $\nu = 0$ or -1, or for $\nu = \alpha$ or $-(\alpha + 1)$, the differential equation is the same, and consequently the solutions are the same. Thus, the eigenvalues, ν, need to be considered only for $\nu \geq -1/2$, or for $\nu \leq -1/2$. For convenience, $\nu \geq -1/2$ is chosen. As a result of this choice and equation (A17), only positive values of ν occur in this analysis.

APPENDIX B

DETERMINATION OF THE EIGENVALUES

It is stated in Chapter III that the eigenvalues are determined by simultaneously solving two continued fraction equations. The purpose of this appendix is to explain how this is accomplished.

Consider the even Dirichlet problem. In order to have solutions of equations (3.21), it is necessary that the eigenvalues satisfy equation (3.26). This equation is an infinite continued fraction containing two unknowns, ν and h. For a derivation of this equation, the reader is referred to Ince's paper[10]. The difficulty here is that the recurrence relations have no starting point. The coefficient subscripts approach both plus and minus infinity. Ince has managed to determine a differential equation which must have the same types of solutions as the ones used here, but with solutions that can be written as series starting with A_0 and going toward A_∞. In this way, the recurrence relations can be written as a matrix equation and a tractable determinant can be identified. Diagonalization of this determinant leads to the continued fraction equation (3.26) which is written here as equation (B1). An eigenvalue equation will be derived in detail later to illustrate this procedure.

$$h = \frac{1 + k^2}{4} + \frac{(2\nu-1)\ (2\nu+3)\ k^2/9}{1 + k^2 - 4h/9}$$

$$\underset{+}{} \frac{16\ (2\nu-3)\ (2\nu+5)\ k^2/225}{1 + k^2 - 4h/25} \underset{+}{} \frac{36\ (2\nu-5)\ (2\nu+7)\ k^2/1225}{1 + k^2 - 4h/49}$$

$$\underset{+}{} \cdot \cdot \cdot \cdot \cdot \cdot \underset{+}{} \frac{4r^2\ (2\nu-2r+1)\ (2\nu+2r+1)\ k^2/(4r^2-1)^2}{1 + k^2 - \dfrac{4h}{(2r+1)^2}} \underset{+}{} \cdot \cdot \tag{B1}$$

and μ is given by

$$\mu = h - \nu(\nu+1)\ k^2 \tag{B2}$$

The conventional notation for a continued fraction expansion is used here, i.e., the term following the lowered plus is added to the denominator of the preceding term. The dummy index r is simply an integer. The continued fraction terminates for ν equal to half integers. For $\nu = 1/2$, h is constant. For $\nu = 3/2$, the equation is a second-order polynomial with two roots; for $\nu = 5/2$, it is a third-order polynomial with three roots; etc. Actually, there are an infinite number of roots for each value of ν. For example, for $\nu = 1/2$, the equation can also be satisfied by equating the entire denominator of the second term to zero; for $\nu = 3/2$, the denominator of the third term is zero; etc.

For $\nu = 1/2$; $h = 0.375$, and $\mu = 0$. This gives one point on a plot of μ versus ν. For $\nu = 3/2$, the lowest root is $h = 1.009$ and $\mu = 0.866$. If values of μ can be determined for ν between 1/2 and 3/2, it will be possible to plot a curve of μ versus ν for the lowest root of equation (B1). This is done in the following manner. For $\nu = 0.6$, a value of h between $h = 0.375$ and $h = 1.009$ is assumed. The first 100 terms of the continued fraction are evaluated for this (ν, h) pair. If the difference between the left hand side and right hand side of the equation is greater then 10^{-2}, the value of h is increased or decreased by 10^{-2} and the continued fraction is evaluated again. This is repeated until either the difference changes signs or the magnitude of the difference is less than 10^{-2}. When this occurs, h is increased or decreased by 10^{-3}. The evaluation is continued until the difference changes signs or the magnitude of the difference is less than 10^{-3}. When this occurs, h is incremented by 10^{-4}. This time, when the difference changes signs or its magnitude is less than 10^{-4}, the last value of h is assumed to be correct. The value of μ is computed and another point is added to the plot of μ versus ν. The value of ν is then increased to 0.7 and the process is repeated again. By incrementing ν in increments of 0.1 between $\nu=0$ and $\nu=9$ and finding μ for each incrementation, a plot of μ_θ^1 versus ν is obtained. The superscript 1 on μ is used to indicate the lowest root, and the subscript θ indicates that it is a root of the θ equation. Table 8 gives the values of μ_θ^1 for $\nu = 0$ to 9, $k^2 = 1/2$, and Figure 29 shows the curve.

Curve μ_θ^2 is obtained by starting with the second root at $\nu = 3/2$. This curve is computed in the same way as curve μ_θ^1. Note that the second root of equation (B1) for $\nu = 1/2$, which can be obtained by equating the denominator of the second term of equation (B1) to zero, can also be obtained by continuing μ_θ^2 back from $\nu = 3/2$ to $\nu = 1/2$.

This is mentioned only to indicate that all of the μ's which are roots for each value of ν can be obtained by using this method. It is seen later that it is not necessary to determine all of the roots for each value of ν.

To be certain that μ_θ^2 is the curve of the second root and not the third or fourth, it is necessary to investigate the area between curves μ_θ^1 and μ_θ^2. The same procedure is used to find h for a fixed value of ν. It is found that there are no values of h lying between curve μ_θ^1 and μ_θ^2 that will satisfy equation (B1). It is also found that there are no roots lying beneath curve μ_θ^1 and thus μ_θ^1 and μ_θ^2 are actually the curves of the first and second roots of equation (B1).

The rest of the roots are found in the same way and are tabulated in Table 8 and plotted in Figure 29.

Some computational difficulties arise in computing the higher order roots. The continued fraction varies rapidly for assumed values of h close to the actual value, and the root is sometimes difficult to find. For example, for a difference in an assumed value of h of as little as 10^{-3}, it is possible for the continued fraction to change from being smaller than h to being greater than h and back again to an even smaller value than the first. When this happens, the computer fails to find the root. Whether or not the root is determined in cases like this depends to a large extent on the initial guess. This accounts for the blanks in Table 8 and also the lack of four place precision in some of the roots.

The choice of 100 terms of the continued fraction was somewhat arbitrary. It was found that the difference in accuracy using 50 terms instead of 100 terms was usually insignificant, so that 50 terms is probably sufficient; however, 100 terms were used to ensure confidence. The extra computer time for 100 terms is insignificant.

The next step in determinining the eigenvalues is to solve the continued fraction equations generated by solutions of equation (3.22). There are two of these given in Chapter III by equations (3.32) and (3.33).

The derivation of equation (3.32) will now be given as an example. All of the other eigenvalue equations used in this paper, except the special one mentioned at the beginning of this appendix, can be derived using the procedure outlined here. The recurrence relation is given by equation (3.30). With some simplification in notation, this set of equations can be written in matrix form as follows:

$$\begin{vmatrix} b_0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & a_4 & b_4 & c_4 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & a_6 & b_6 & c_6 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & a_8 & b_8 & c_8 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & a_{10} & b_{10} & c_{10} & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & a_{12} & b_{12} & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{14} & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \begin{vmatrix} B_0 \\ B_2 \\ B_4 \\ B_6 \\ B_8 \\ B_{10} \\ B_{12} \\ B_{14} \\ \cdot \\ \cdot \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \end{vmatrix} \tag{B3}$$

In order to have a non-trivial solution it is necessary that the determinant be equal to zero. Thus the eigenvalue equation is

$$\begin{Vmatrix} b_0 & c_0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 & \cdot \\ 0 & a_4 & b_4 & c_4 & 0 & 0 & 0 & \cdot \\ 0 & 0 & a_6 & b_6 & c_6 & 0 & 0 & \cdot \\ 0 & 0 & 0 & a_8 & b_8 & c_8 & 0 & \cdot \\ 0 & 0 & 0 & 0 & a_{10} & b_{10} & c_{10} & \cdot \\ 0 & 0 & 0 & 0 & 0 & a_{12} & b_{12} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{Vmatrix} = 0 \qquad \text{(B4)}$$

For the purpose of the derivation assume that the matrix in equation (B4) can be truncated as a 7×7 matrix. Starting at the lower right hand corner, use standard techniques to diagonalize the matrix. First, multiply column seven by $\frac{a_{12}}{b_{12}}$ and subtract from column six. This creates a zero in column six, row seven. Now multiply row seven by $\frac{c_{10}}{b_{12}}$ and subtract from row six. This creates a zero in column seven, row six. The matrix now has the form

$$\begin{vmatrix} b_0 & c_0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 \\ 0 & a_4 & b_4 & c_4 & 0 & 0 & 0 \\ 0 & 0 & a_6 & b_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & a_8 & b_8 & c_8 & 0 \\ 0 & 0 & 0 & 0 & a_{10} & (b_{10} - \frac{c_{10}\, a_{12}}{b_{12}}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{12} \end{vmatrix} \qquad \text{(B5)}$$

Note that the upper left 6 x 6 matrix has the same form as the previous 7 x 7 matrix. Repeat the process on this matrix, and on the succeeding ones until the entire matrix is diagonalized. The final form of the matrix is diagonal with each term a continued fraction. The term in the upper left hand corner is

$$b_0 - \frac{a_2\, c_0}{b_2 - \frac{a_4\, c_2}{b_4 - \frac{a_6\, c_4}{b_6 - \frac{a_8\, c_6}{b_8 - \frac{a_{10}\, c_8}{b_{10} - \frac{a_{12}\, c_{10}}{b_{12} - \cdots}}}}}} \tag{B6}$$

The second term is just the denominator of the second term of equation (B6). It is

$$b_2 - \frac{a_4\, c_2}{b_4 - \frac{a_6\, c_4}{b_6 - \frac{a_8\, c_6}{b_8 - \frac{a_{10}\, c_8}{b_{10} - \frac{a_{12}\, c_{10}}{b_{12} - \cdots}}}}} \tag{B7}$$

The rest of the terms follow this same pattern, i.e., the third term is just the denominator of the second term of equation (B7), etc. Although the matrix was truncated for the purpose of the derivation, the pattern of the continued fraction for the infinite matrix is obvious from the above equations. The eigenvalue equation is obtained by requiring that the first term be equal to zero. Writing out the a_m, b_m, and c_m terms, rearranging and simplifying, the eigenvalue equation can be written as equation (3.32) which is written below as equation (B8).

$$\eta = \cfrac{-\,(\nu^2-1)(\nu)(\nu+2)\ k'^4/8}{2 - k'^2 - \eta/4 \quad - \cfrac{(\nu^2-9)(\nu-2)(\nu+4)\ k'^4/(16)^2}{2 - k'^2 - \eta/16}}$$

$$- \cfrac{(\nu^2-25)(\nu-4)(\nu+6)\ k'^4/(48)^2}{2 - k'^2 - \eta/36} \quad - \cdots\cdot$$

$$- \cfrac{(\nu^2-(2r-1)^2)(\nu-2r+2)(\nu+2r)\ k'^4/(8r(r-1))^2}{2 - k'^2 - \eta/4r^2} \quad - \cdots\cdot \tag{B8}$$

and μ is given by

$$\mu = 1/2\ (-\eta + \nu(\nu+1)\ k'^2) \tag{B9}$$

The continued fraction terminates for integer values of ν. Using the integers as starting points, curves of μ versus ν are computed using the same procedure as before. These curves are labeled μ_ϕ^1, μ_ϕ^3, μ_ϕ^5, etc. The roots for $k'^2 = 1/2$ are tabulated in Table 9 and the curves are drawn in Figure 29. Equation (3.33) is

$$\eta = \nu(\nu+1)\ k'^2/2 + 2 - k'^2 - \cfrac{(\nu^2-4)(\nu-1)(\nu+3)\ k'^4/(6)^2}{2 - k'^2 - \eta/9}$$

$$- \cfrac{(\nu^2-16)(\nu-3)(\nu+5)\ k'^4/(30)^2}{2 - k'^2 - \eta/25} \quad - \cdots\cdot$$

$$- \cfrac{(\nu^2-4r^2)(\nu-2r+1)(\nu+2r+1)\ k'^4/(2(4r^2-1))^2}{2 - k'^2 - \eta/(2r+1)^2} \quad - \cdots\cdot \tag{B10}$$

and μ is given by equation (B9). This continued fraction also terminates for integer values of ν. The roots are determined as before and are tabulated in Table 9 and the curves are drawn in Figure 29. These curves are labeled μ_ϕ^2, μ_ϕ^4, μ_ϕ^6, etc.

The intersections of the θ and ϕ curves in Figure 29 give the (ν, μ) pairs that satisfy both equations. These are the eigenvalues of the even Dirichlet problem and are tabulated in Table 1. Even though the eigenvalues are determined graphically, they should be quite accurate. Accuracy is achieved by plotting each intersection on a scale large enough to use the four place accuracy of μ. It is felt that the eigenvalues determined by using this graphical procedure are correct to three places for most cases; however, accuracy is only claimed to $\pm 5 \times 10^{-3}$. Better accuracy can be achieved, if desired, by using finer increments of ν.

From the pattern formed by the curves, it is evident why all of the eigenvalue pairs for each continued fraction are not needed. The family of θ curves is moving upward and the family of ϕ curves is moving downward as the order of the root is increased. The area beneath the μ_θ^1 curve and the area above the μ_ϕ^1 curve contain roots of each continued fraction but no roots that can simultaneously satisfy the θ and ϕ equations. Thus it is not necessary to solve the continued fraction equations in these regions.

It is seen from the curves that the number of eigenvalues is increasing as ν increases. If the integer n is defined as it is in Chapter III, $n - 1/2 \leq \nu \leq n + 1/2$, there are $n + 1$ eigenvalues for each value of ν. This agrees with the number of eigenvalues for the even Lamé polynomials. It is interesting to notice the slope of the curves. The slope of all the curves is decreasing as $\nu = 0$ is approached. Actually, the slope will be zero at $\nu = -1/2$, and the curves will be symmetrical around a vertical line at $\nu = -1/2$. This is due to the symmetry mentioned in Appendix A.

Depending on the form of the eigenfunctions, there are two continued fraction equations for the θ eigenvalues of the odd Dirichlet problem. Equation (3.43) is

$$\eta = -\nu(\nu+1)\,k^2/2 + 2 - k^2 - \cfrac{(\nu^2-4)(\nu-1)(\nu+3)\,k^4/(6)^2}{2 - k^2 - \eta/9 - \cfrac{(\nu^2-16)(\nu-3)(\nu+5)\,k^4/(30)^2}{2 - k^2 - \eta/25 - \cdots}}$$

$$- \cfrac{(\nu^2-4r^2)(\nu-2r+1)(\nu+2r+1)\,k^4/(2(4r^2-1))^2}{2 - k^2 - \eta/(2r+1)^2 - \cdots} \tag{B11}$$

and μ is given by

$$\mu = 1/2\,(\eta - \nu(\nu+1)\,k^2) \tag{B12}$$

The roots are labeled μ_θ^1, μ_θ^3, μ_θ^5, etc. Their negative values are tabulated in Table 10 and the curves are shown in Figure 30.

Equation (3.45) is

$$\eta = 4(2-k^2) - \cfrac{(\nu^2-9)(\nu-2)(\nu+4)k^4/(8)^2}{2 - k^2 - \eta/16 - \cfrac{(\nu^2-25)(\nu-4)(\nu+6)k^4/(48)^2}{2 - k^2 - \eta/36}}$$

$$- \cdots - \cfrac{(\nu^2-(2r+1)^2)(\nu-2r)(\nu+2r+2)\,k^4/(4r(2r+2))^2}{2 - k^2 - \eta/(2r+2)^2 - \cdots} \tag{B13}$$

and μ is given by equation (B12). These roots are labeled μ_θ^2, μ_θ^4, μ_θ^6, etc. Their negatives are tabulated in Table 10 and the curves are shown in Figure 30. The continued fraction equations for the ϕ eigenvalues are the same as equations (B11) and (B13) with k^2 replaced by k'^2. For $k^2 = k'^2 = 1/2$, the equations are the same, and the only difference is in the sign of μ.

$$\mu = 1/2\ (-n + \nu(\nu+1)\ k'^2) \tag{B14}$$

Thus the family of ϕ curves is the mirror image of the family of θ curves. The eigenvalues satisfying both sets of equations correspond to integer values of ν, and the eigenfunctions are Lamé polynomials. There are n eigenvalues for each $\nu = n$, so that the Dirichlet problem has a total of $2n + 1$ eigenvalues for each value of ν. This is indicated by the grouping of the eigenvalues in Table 1.

Accuracy is better for the odd Dirichlet problem than for the even Dirichlet problem, since it is known that the intersections occur when ν is an integer, and the eigenvalue μ has been computed for integer values of ν. Thus no graphical interpolation is necessary. This remark also applies to the eigenvalues of the even Neumann problem.

The even Neumann problem is also similar to the odd Dirichlet problem in that the continued fractions for the θ and ϕ solutions are the same. The continued fractions for the ϕ equation are equations (B8) and (B10) which have already been solved. The curves for the θ equation are the mirror image of the ϕ curves. The roots of the ϕ equations and the negative roots of the θ equations are tabulated in

Table 9. The curves are shown in Figure 31. The eigenvalues correspond to $\nu = n$, an integer; and there are $n + 1$ μ's for each ν. They are tabulated in Table 3.

The continued fraction for the θ equation of the odd Neumann problem is given by equation (B1). This equation has already been solved and the roots tabulated. The curves are shown in Figure 32. The continued fractions for the ϕ equations are equations (B11) and (B13) with k^2 replaced by k'^2. The roots are tabulated in Table 10, and the curves are shown in Figure 32. The eigenvalues are tabulated in Table 3. There are n eigenvalues for each ν, where $n - 1/2 \leq \nu \leq n + 1/2$, making a total of $2n + 1$ eigenvalues for each ν of the Neumann problem.

The lowest roots of the even Dirichlet problem and the odd Neumann problem for $k^2 = 0.1$ and $k^2 = 0.9$ were also determined. The roots are tabulated in Tables 11, 12, and 13, and the curves are drawn in Figures 33 and 34. The eigenvalues are given in Table 5.

Values of ν and h for the cases when the continued fractions in this appendix terminate have been computed by Ince[11] and Arscott[14]. They are in agreement with the numbers presented here. Ince has also discussed some properties of the continued fractions, such as convergence and asymptotic characteristics. The convergence property is the main reason for the use of the variable η in some of the continued fractions. The reader is cautioned that Ince has used differential equations different from the ones used in this paper. A simple change of variables converts his equations into the equations used here; however, a direct comparison of the eigenvalues is not easy.

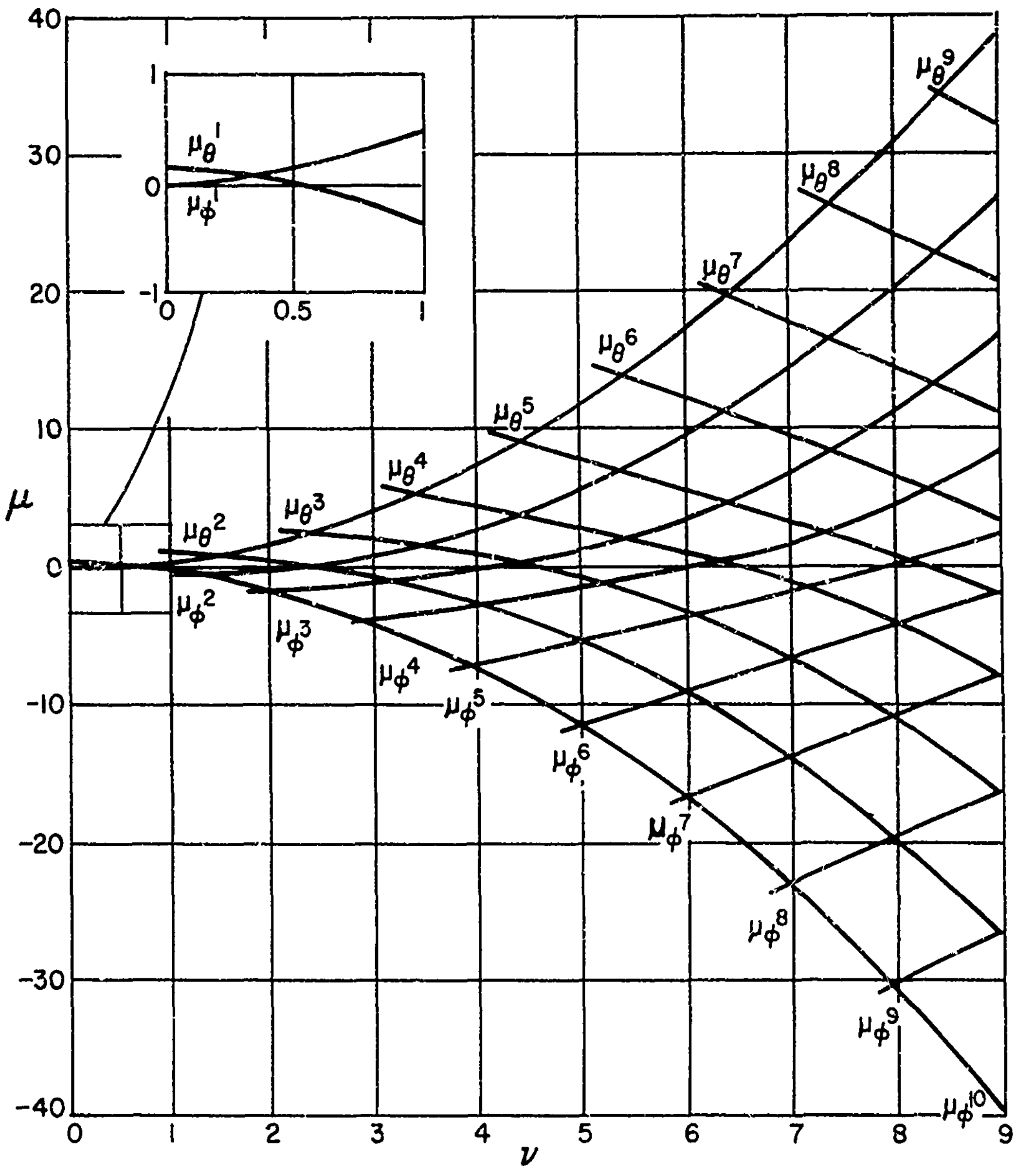

Fig. 29. Eigenvalues for the even Dirichlet problem

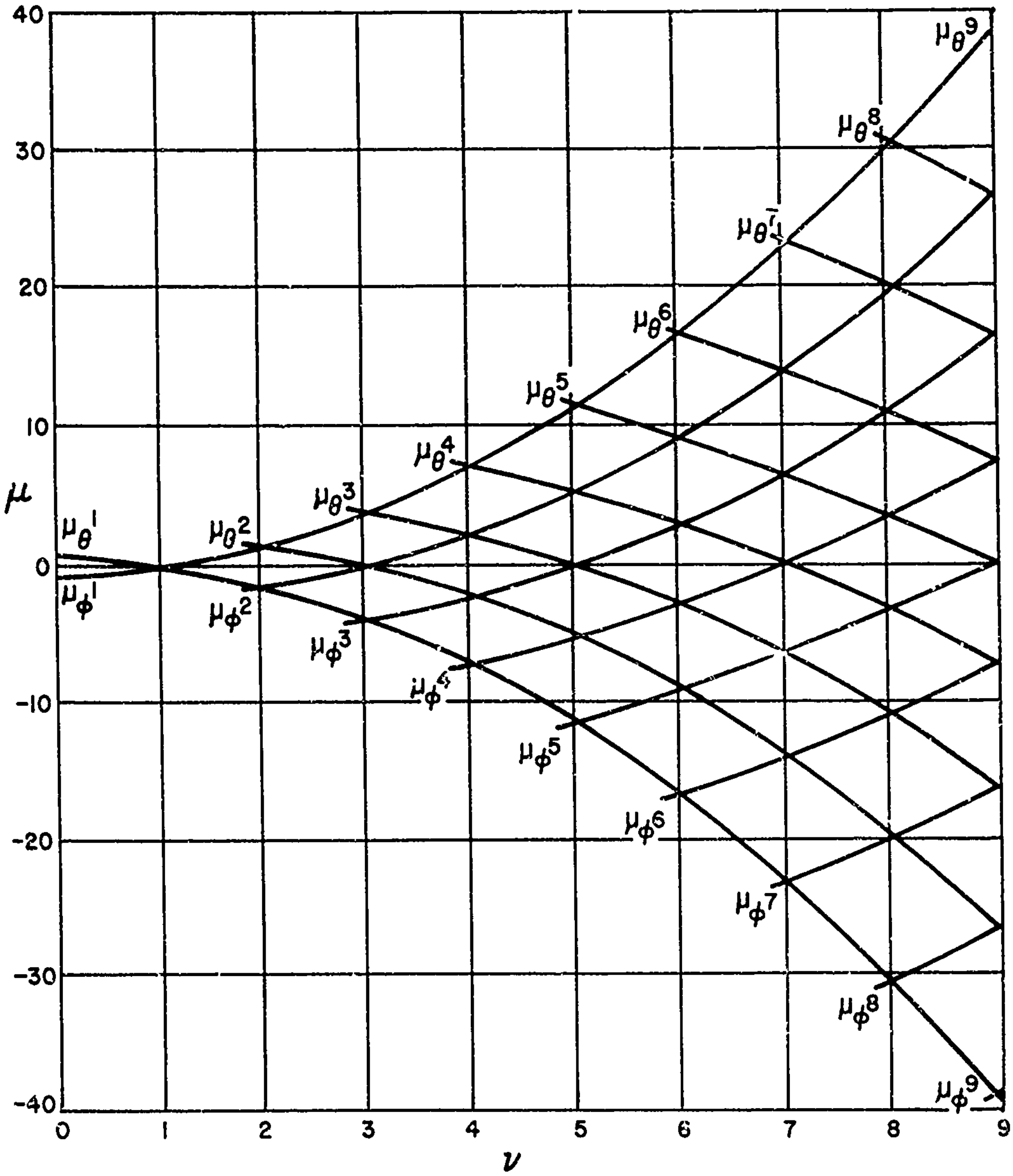

Fig. 30. Eigenvalues for the odd Dirichlet problem

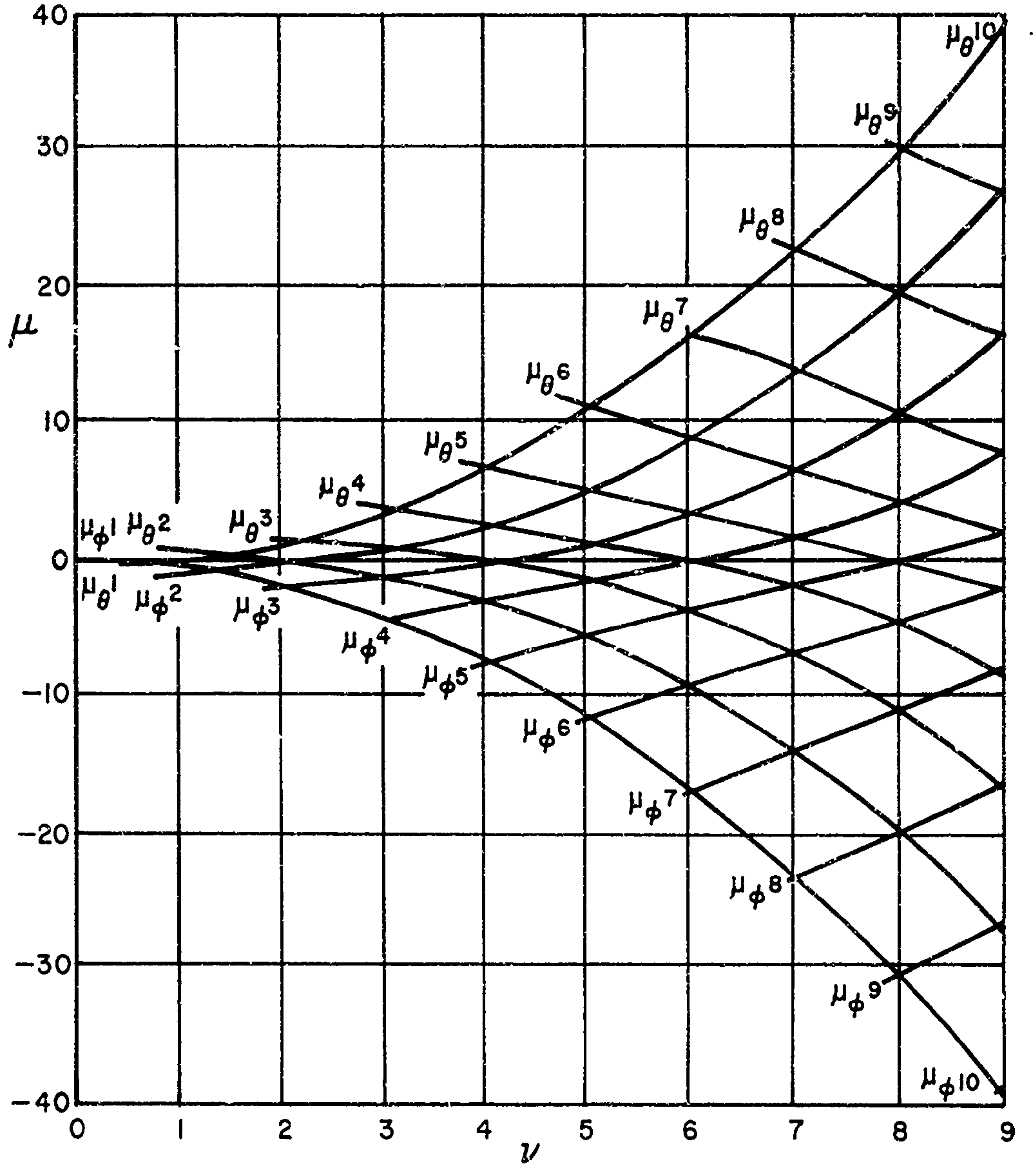

Fig. 31. Eigenvalues for the even Neumann problem

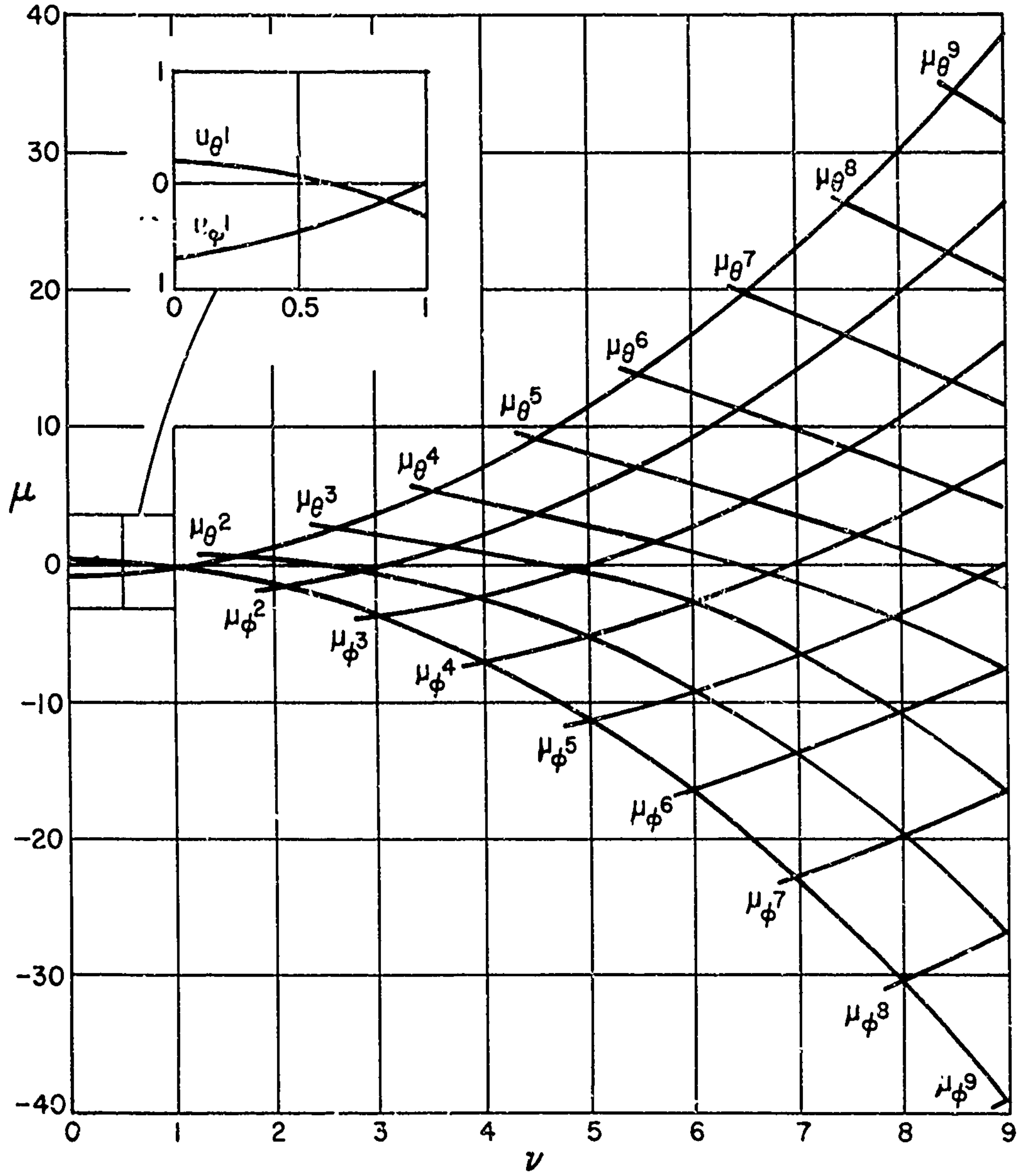

Fig. 32. Eigenvalues for the odd Neumann problem

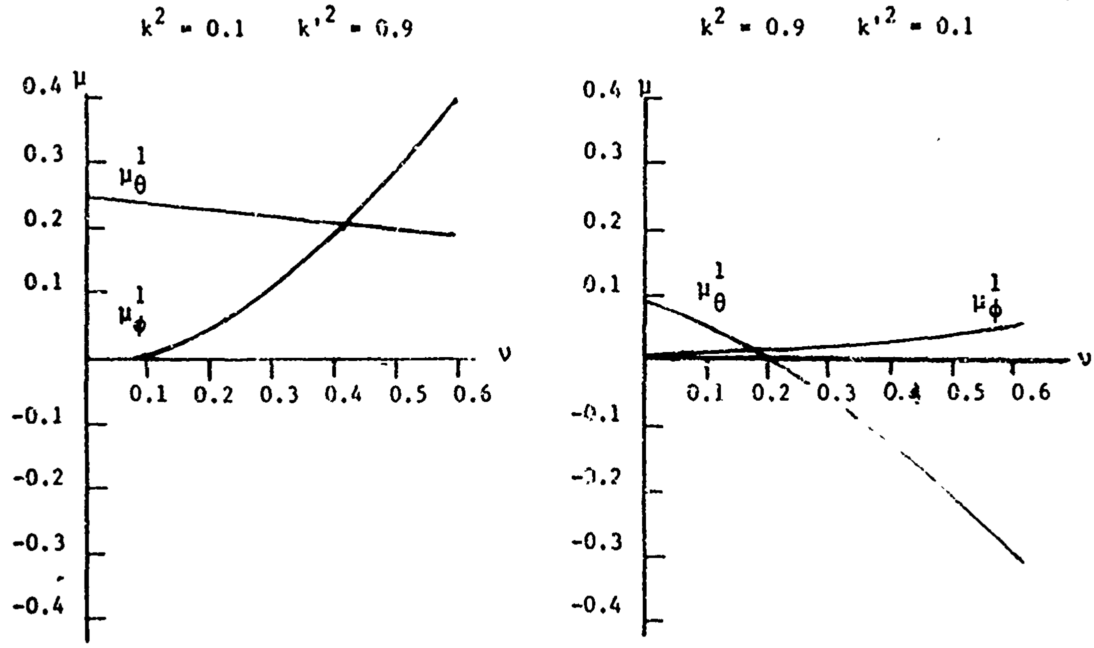

Fig. 33. Eigenvalues for the even Dirichlet problem

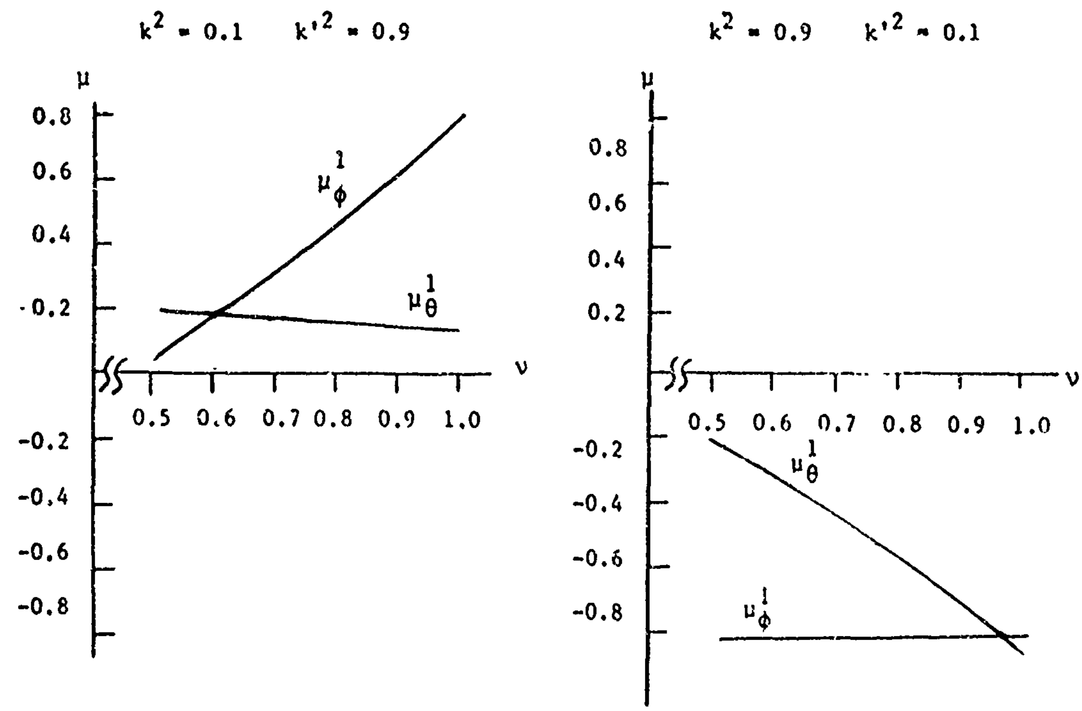

Fig. 34. Eigenvalues for the odd Neumann problem

Table 5 - Eigenvalues of the θ Equation for the Even Dirichlet Problem and the Odd Neumann Problem

ν	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
0.0	0.1794								
0.1	0.1541								
0.2	0.1236								
0.3	0.0881								
0.4	0.0470								
0.5	0.0000								
0.6	- 0.0531								
0.7	- 0.1127								
0.8	- 0.1792								
0.9	- 0.2528								
1.0	- 0.3341	1.1915							
1.1	- 0.4232	1.1314							
1.2	- 0.5207	1.0686							
1.3	- 0.6268	1.0033							
1.4	- 0.7418	0.9360							
1.5	- 0.8860	0.8659							
1.6	- 0.9996	0.7937							
1.7	- 1.1425	0.7188							
1.8	- 1.2958	0.6414							
1.9	- 1.4586	0.5609							
2.0	- 1.6313	0.4773							
2.1	- 1.8140	0.3902	3.0856						
2.2	- 2.0068	0.2992	2.9795						
2.3	- 2.2097	0.2042	2.8708						
2.4	- 2.4227	0.1047	2.7595						
2.5	- 2.6458	0.0000	2.6457						
2.5	- 2.8789	- 0.1097	2.5300	6.7376					
2.7	- 3.1222	- 0.2253	2.4122	6.6053					
2.8	- 3.3756	- 0.3475	2.2927	6.4696					
2.9	- 3.6390	- 0.4763	2.1715	6.3305					
3.0	- 3.9126	- 0.6123	2.0487	6.1882					
3.1	- 4.1962	- 0.7559	1.9245	6.0427					
3.2	- 4.4898	- 0.9076	1.7991	5.8941					
3.3	- 4.7935	- 1.0677	1.6723	5.7427					
3.4	- 5.1073	- 1.2366	1.5444	5.5884					
3.5	- 5.4312	- 1.4146	1.4150	5.4312					
3.6	- 5.7650	- 1.6020	1.2842	5.2714					
3.7	- 6.1090	- 1.799	1.1519	5.1092	10.7300				
3.8	- 6.4629	- 2.0060	1.0177	4.9446	10.5527				
3.9	- 6.8268	- 2.2226	0.8814	4.7777	10.3718				
4.0	- 7.2008	- 2.4496	0.7428	4.6087	10.1879				
4.1	- 7.5848		0.6015	4.4377	10.0097				
4.2	- 7.9789	- 2.9339	0.4569	4.2651	9.8103				
4.3	- 8.3829	- 3.1913	0.3087	4.0907	9.6109				
4.4	- 8.7970	- 3.4593	0.1566	3.9149	9.4204				
4.5	- 9.2211	- 3.738	0.0000	3.7379	9.2211				
4.6	- 9.6552	- 4.0260	-0.1618	3.5597	9.0189				
4.7	-10.0993	- 4.3247	-0.3291	3.3807	8.8137				
4.8	-10.5534		-0.5023	3.2077	8.6061				
4.9	-11.0175	- 4.9533	-0.6822	3.0199	8.3967				
5.0	-11.4916	- 5.2829	-0.8619	2.8389	8.1830				
5.1	-11.9758	- 5.6627	-1.0635	2.6573	7.9679	14.9590			
5.2	-12.4699	- 5.9730	-1.2661	2.4750	7.7507	14.7210			
5.3	-12.9740	- 6.3332	-1.4769	2.2926	7.5306	14.4919			
5.4	-13.4882	- 6.7037	-1.6969	2.1093	7.3089	14.2537			
5.5	-14.0124	- 7.0844	-1.9253	1.9260	7.0852	14.0124			
5.6	-14.5466	- 7.4753	-2.1641	1.7420	6.8593	13.7682			
5.7	-15.0908	- 7.8763	-2.4125	1.5565	6.6320	13.5213			
5.8	-15.6449	- 8.2873	-2.6706	1.3693	6.4029	13.2712			
5.9	-16.2091	- 8.7084	-2.9392	1.1818	6.1732	13.0185			
6.0	-16.7833	- 9.1399	-3.2174	0.9917	5.9419	12.7633	21.199		
6.1	-17.3675	- 9.5810	-3.5072	0.8010	5.7085	12.5050	20.930		
6.2	-17.9617	-10.0325	-3.8061	0.6048	5.4751	12.2446	20.651		
6.3	-18.5659	-10.4939	-4.1164	0.4070	5.2408	11.9819	20.3672		
6.4	-19.1801	-10.9657	-4.4366	0.2036	5.0060	11.7158	20.0874		
6.5	-19.8043	-11.4472	-4.7673	0.0000	4.7707	11.4485	19.8045		
6.6	-20.4385	-11.9390	-5.1091	-0.2115		11.1792	19.5183		
6.7	-21.0827	-12.4408	-5.4611	-0.4265	4.2974	10.9075	19.2293		
6.8	-21.7370	-12.9525	-5.8254	-0.6487		10.6322	18.9379		
6.9	-22.4011	-13.4746	-6.1987	-0.8792	3.8251	10.3557	18.6431		
7.0	-23.0753	-14.0066	-6.5819	-1.1131	3.5849	10.0772	18.3454		
7.1	-23.7579	-14.5483	-6.9771	-1.3550		9.7989	18.0451	27.87	
7.2	-24.4537	-15.1003	-7.3815	-1.6053	3.1167	9.5196	17.7416	27.56	
7.3	-25.1579	-15.6627	-7.7960	-1.8657	2.8819	9.2352	17.4353	27.23	
7.4	-25.8722	-16.2347	-8.2201	-2.1353	2.6478	8.9546	17.1279	26.91	
7.5	-26.5965	-16.8169	-8.6582	-2.4129		8.6694	16.8177	26.5969	
7.6	-27.3308	-17.4088	-9.1048	-2.6949	2.1753	8.3811	16.5034	26.2696	
7.7	-28.0750	-18.0109	-9.5608	-2.9879	1.9383	8.0966	16.1861	25.9387	
7.8	-28.8292	-18.6232	-10.0283	-3.2959	1.6932	7.8114	15.8699	25.6057	
7.9	-29.5935	-19.2454	-10.5049	-3.6173	1.4603	7.5149	15.5488		
8.0	-30.3677	-19.8779	-10.9905	-3.9277	1.2211	7.2176	15.2292	24.9283	
8.1	-31.1519	-20.5199	-11.4896	-4.2698	0.9570	6.9422	14.9050	24.5871	
8.2	-31.9462	-21.1722	-11.9963	-4.6150	0.7237	6.6570	14.5738	24.2414	
8.3	-32.7504	-21.8345	-12.5155		0.4747	6.3528	14.2496	23.8972	
8.4	-33.5646	-22.5072		-5.3459	0.2286	6.0796	13.9214		
8.5	-34.3889	-23.1893	-13.5839	-5.7369	0.0000	5.7674	13.5861	23.1908	34.39
8.6	-35.2231	-23.8814	-14.1271	-6.1194	-0.2705	5.4662	13.2389	22.8382	34.03
8.7	-36.0674	-24.5840	-14.6889	-6.5250	-0.5654	5.1991	12.9328	22.4826	33.65
8.8	-36.9216	-25.2961	-15.2553	-6.9455	-0.7995	4.9340		22.1193	33.28
8.9	-37.7858	-26.0188	-15.8327	-7.3579	-1.0618	4.6367	12.2436	21.7501	32.89
9.0	-38.6601	-26.7510	-16.4260	-7.8065	-1.3339	4.3595	11.8892	21.4041	32.51

Table 9 - Eigenvalues of the φ Equation for the Even Dirichlet and Neumann Problems. (Eigenvalues of the θ Equation for the Even Neumann Problem are Given by $-\mu$)

ν	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
0.0	0.0000								
0.1	0.0253								
0.2	0.0555								
0.3	0.0907								
0.4	0.1313								
0.5	0.1774								
0.6	0.2293								
0.7	0.2872								
0.	0.3514								
0.9	0.4222	-0.5328							
1.0	0.5000	-0.5000							
1.1	0.5851	-0.4645	-2.3660						
1.2	0.6777	-0.4261	-2.3014						
1.3	0.7783	-0.3848	-2.2348						
1.4	0.8873	-0.3404	-2.1665						
1.5	1.0048	-0.2927	-2.0967						
1.6	1.1313	-0.2416	-2.0256						
1.7	1.2671	-0.1870	-1.9534						
1.8	1.4123	-0.1287	-1.8803						
1.9	1.5672	-0.0664	-1.8065						
2.0	1.7320	0.0000	-1.7320						
2.1	1.9069	0.0707	-1.6570						
2.2	2.0921	0.1461	-1.5814						
2.3	2.2875	0.2263	-1.5052						
2.4	2.4934	0.3117	-1.4284						
2.5	2.7098	0.4025	-1.3510						
2.6	2.9367	0.4991	-1.2727						
2.7	3.1741	0.6018	-1.1936						
2.8	3.4220	0.7108	-1.1135						
2.9	3.6805	0.8266	-1.0321	-4.0773					
3.0	3.9495	0.9495	-0.9495	-3.9495					
3.1	4.2289	1.0798	-0.8653	-3.8205					
3.2	4.5188	1.2180	-0.7793	-3.6907					
3.3	4.8192	1.3644	-0.6913	-3.5602					
3.4	5.1299	1.5193	-0.6012	-3.4293					
3.5	5.4510	1.6831	-0.5086	-3.2983					
3.6	5.7825	1.8561	-0.4133	-3.1672					
3.7	6.1242	2.0386	-0.3151	-3.0362					
3.8	6.4762	2.2308	-0.2137	-2.9056					
3.9	6.8385	2.4331	-0.1087	-2.7755	-7.3869				
4.0	7.2110	2.6457	0.0000	-2.6457	-7.2111				
4.1	7.5937	2.8687	0.1128	-2.5166	-7.0330				
4.2	7.9866	3.1023	0 2300	-2.3880	-6.8528				
4.3	8.3896	3.3466	0.3520	-2.2599	-6.6708				
4.4	8.8028	3.6018	0.4791	-2.1323	-6.4871				
4.5	9.2261	3.8679	0.6116	-2.0050	-6.3021				
4.6	9.6595	4.1449	0.7499	-1.8781	-6.1157				
4.7	10.1030	4.4330	0.8944	-1.7511	-5.9284				
4.8	10.5566	4.7319	1.0455	-1.6241	-5.7403				
4.9	11.0203	5.0420	1.2035		-5.5517	-11.7138			
5.0	11.4941	5.3629	1.3688	-1.3688	-5.3629	-11.4941			
5.1	11.9779	5.6948	1.5418	-1.2401	-5.1740	-11.2717			
5.2	12.4718	6.0375	1.7230	-1.1104	-4.9854	-11.0466			
5.3	12.9757	6.3911	1.9127	-0.9794	-4.7972	-10.8190			
5.4	13.4896	6.7554	2.1112	-0.8468	-4.6096	-10.5889			
.5	14.0136	7.1305	2.3190	-0.7123	-4.4228	-10.3566			
5.6	14.5476	7.5162	2.5363	-0.5756	-4.2372	-10.1220			
5.7	15.0916	7.9126	2.7635	-0.4363	-4.0526	- 9.8855			
5.8	15.6457	8.3196	3.0009	-0.2942	-3.8694	- 9.6470			
5.9	16.2098	8.7371	3.2487	-0.1489	-3.6875	- 9.4068	-17.0458		
6.0	16.7839	9.1651	3.5072	0.0000	-3.5071	- 9.1652	-16.7839		
6.1	17.3680	9.6036	3.7765	0.1528	-3.3282	- 8.9221	-16.5190		
6.2	17.9621	10.0524	4.0569	0.3099	-3.1507	- 8.6778	-16.2512		
6.3	18.5662	10.5116	4.3485	0.4717	-2.9747	-8.4326	-15.9807		
6.4	19.1804	10.9812	4.6513	0.6386	-2.7999	-8.1867	15.7073		
6.5	18.8046	11.4610	4.9655	0.8110	-2.6263	-7.9402	-15.4313		
6.6	20.4387	11.9511	5.2912	0.9863	-2.4538	-7.6934	-15.1527		
6.7	21.0829	12.4515	5.6282	1.1739	-2.2821	-7.4465	-14.8715		
6.8	21.7371	12.9621	5.9767	1.3652	-2.1110	-7.1999	-14.5879		
6.9	22.4013	13.4828	6.3367	1.5637	-1.9404	-6.9536	-14.3020	-23.3792	
7.0	23.0755	14.0138	6.7080	1.7697	-1.7697	-6.7081	-14.0138	-23.0755	
7.1	23.7597	14.5549	7.09[illegible]	1.9838	-1.5990	-6.4634	-13.7235	-22.7688	
7.2	24.4539	15.1062	7.4847	2.2063	-1.4278	-6.2198	-13.4312	-22.4592	
7.3	25.1581	15.6675	7.8899	2.4376	-1.2559	-5.9776	-13.1370	-22.1465	
7.4	25.8723	16.2390	8.3063	2.6781	-1.0827	-5.7370	-12.8409	-21.8310	
7.5	26.5965	16.8206	8.7339	2.9283	-0.9[illegible]1	-5.4980	-12.5433	-21.5126	
7.6	27.3306	17.4123	9.1725	3.1883	-0.7317	-5.2610	-12.2442	-21.1914	
7.7	28.0750	18.0141	9.6221	3.4587	-0.5531	-5.0261	-11.9437	-20.8675	
7.8	28.8292	18.6259	10.0825	3.7396	-0.3718	-4.7933	-11.6421	-20.5409	
7.9	29.5935	19.2478	10.5539	4.0315	-0.1876	-4.5627	-11.3395	-20.1173	-30.7131
8.0	30.3677	19.8799	11.0360	4.3344	0.0000	-4.3344	-11.0361	-19.8799	-30.3678
8.1	31.1519	20.5219	11.5288	4.6486	0.1914	-4.1084	-10.7320	-19.5457	-30.0195
8.2	31.9461	21.1740	12.0323	4.9743	0.3871	-3.8846	-10.4276	-19.2091	-29.6680
8.3	32.7504	21.8361	12.5464	5.3116	0.8574	-3.6629	-10.1229	-18.8701	-29.3137
8.4	33.5646	22.5083	13.0711	5.6607	0.7929	-3.4433	- 9.8183	-18.5288	-28.9563
8.5	34.3888	23.1905	13.6062	6.0216	1.004	-3.2256	- 9.5139	-18.1853	-28.5960
8.6	35.2231	23.8827	14.1519	6.3944	1.2208	-3.0096		-17.8398	-28.2328
8.7	36.0673	24.5850	14.7079	6.7791	1.4443	-2.7951	- 8.9070	-17.4922	-27.8668
8.8	36.9216	25.2973	15.2744	7.1757	1.6747	-2.5819	- 8.6049	-17.1427	-27.4980
8.9	37.7858	26.0196	15.8512	7.5841	1.9125	-2.3696	- 8.3040	-16.7475	-27.1264
9.0	38.6501	26.7520	16.4383	8.0044	2.1581	-2.1580	- 8.0046	-16.3867	-26.7522

ν	μ_{10}
9.0	-38.6601

Table 10 - Eigenvalues of the ϕ Equation for the Odd Dirichlet and Neumann Problems. (Eigenvalues of the θ Equation for the Odd Dirichlet Problem are Given by $-\mu$)

ν	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
0.0	-0.7178								
0.1	-0.6789								
0.2	-0.6329								
0.3	-0.5797								
0.4	-0.5193								
0.5	-0.4515								
0.6	-0.3763								
0.7	-0.2936								
0.8	0.2034								
0.9	-0.1055								
1.0	0.0000								
1.1	0.1133								
1.2	0.2346								
1.3	0.3638								
1.4	0.5011								
1.5	0.6465								
1.6	0.8003								
1.7	0.9624								
1.8	1.1330								
1.9	1.3122	-1.6168							
2.0	1.5000	-1.5000							
2.1	1.6965	-1.3775							
2.2	1.9020	-1.2492							
2.3	2.1163	-1.1150							
2.4	2.3396	-0.9747							
2.5	2.5720	-0.8284							
2.6	2.8136	-0.6757							
2.7	3.0644	-0.5167							
2.8	3.3246	-0.3511							
2.9	3.5940	-0.1750	-4.0173						
3.0	3.8729	0.0000	-3.8729						
3.1	4.1613	0.1858	-3.7245						
3.2	4.4592	0.3787	-3.5719						
3.3	4.7667	0.5787	-3.4150						
3.4	5.0838	0.7860	-3.2538						
3.5	5.4105	1.0008	-3.0882						
3.6	5.7470	1.2231	-2.9180						
3.7	6.0931	1.4530	-2.7433						
3.8	6.4491	1.6908	-2.5639						
3.9	6.8148	1.9366	-2.3796	-7.3712					
4.0	7.1930	2.1903	-2.1904	-7.1904					
4.1	7.5757	2.4523	-1.9962	-7.0062					
4.2	7.9709	2.7226	-1.7968	-6.8187					
4.3	8.3759	3.0013	-1.5921	-6.6278					
4.4	8.7909	3.2885	-1.3820	-6.4337					
4.5	9.2158	3.5843	-1.1664	-6.2361					
4.6	9.6506	3.8888	-0.9451	-6.0350					
4.7	10.0952	4.2022	-0.7179	-5.8306					
4.8	10.5499	4.5244	-0.4848	-5.6627					
4.9	11.0145	4.8557	-0.2455	-5.4112	-11.7101				
5.0	11.4890	5.1961	0.0000	-5.1961	-11.4891				
5.1	11.9735	5.5456	0.2519	-4.9775	-11.2651				
5.2	12.4679	5.9043	0.5109	-4.7551	-11.0379				
5.3	12.9723	6.2724	0.7755	-4.5288	-10.8078				
5.4	13.4867	6.6498	1.0475	-4.2988	-10.5747				
5.5	14.0111	7.0367	1.3265	-4.0647	-10.3386				
5.6	14.5454	7.4330	1.6127	-3.8267	-10.0996				
5.7	15.0897	7.8389	1.9061	-3.5844	-9.8576				
5.8	15.6440	8.2544	2.2070	-3.3379	-9.6127				
5.9	16.2083	8.6795	2.5154	-3.0870	-9.3650	-17.0450			
6.0	16.7826	9.1142	2.8315	-2.8316	-9.1143	-16.7828			
6.1	17.3669	9.5587	3.1555	-2.5715	-8.8607	-16.5175			
6.2	17.9611	10.0128	3.4874	-2.3067	-8.6042	-16.2492			
6.3	18.5654	10.4768	3.8274	-2.0370	-8.3448	-15.9780			
6.4	19.1796	10.9505	4.1757	-1.7623	-8.0824	-15.7039			
6.5	19.8039	11.4341	4.5323	-1.4324	-7.8170	-15.4268			
6.6	20.4381	11.9275	4.8973	-1.1972	-7.5487	-15.1470			
6.7	21.0824	12.4307	5.2709	-0.9064	-7.2773	-14.8643			
6.8	21.7366	12.9439	5.6532	-0.6101	-7.0028	-14.5789			
6.9	22.4008	13.4669	6.0443	-0.3081	-6.7251	-14.2908	-23.3790		
7.0	23.0751	14.0000	6.4442	0.0000	-6.4443	-14.0000	-23.0753		
7.1	23.7593	14.5427	6.8531	0.3140	-6.1602	-13.7065	-22.7685		
7.2	24.4535	15.0954	7.2711	0.6344	-5.8727	-13.4104	-22.4587		
7.3	25.1578	15.6582	7.6981	0.9612	-5.5817	-13.1117	-22.1459		
7.4	25.8720	16.2309	8.1344	1.2945	-5.2872	-12.8104	-21.8302		
7.5	26.5963	16.8135	8.5800	1.6345	-4.9892	-12.5066	-21.5115		
7.6	27.3305	17.4061	9.0340	1.9813	-4.6874	-12.2002	-21.1900		
7.7	28.0747	18.0087	9.4992	2.3352	-4.3817	-11.8913	-20.8657		
7.8	28.8289	18.6212	9.9730	2.6963	-4.0722	-11.5798	-20.5387		
7.9	29.5932	19.2437	10.4563	3.0647	-3.7585	-11.2658	-20.2090	-30.7131	
8.0	30.3674	19.8762	10.9492	3.4405	-3.4406	-10.9493	-19.8764	-30.3678	
8.1	31.1517	20.5187	11.4516	3.8240	-3.1184	-10.6302	-19.5414	-30.0194	
8.2	31.9459	21.1712	11.9638	4.2152	-2.7918	-10.3086	-19.2036	-29.6680	
8.3	32.7501	21.8337	12.4856	4.6143	-2.4605	-9.9844	-18.8634	-29.3135	
8.4	33.5644	22.5061	13.0172	5.0215	-2.1245	-9.6576	-18.5206	-28.9561	
8.5	34.3886	23.1886	13.5586	5.4367	-1.7836	-9.3282	-18.1753	-28.5958	
8.6	35.2228	23.8811	14.1097	5.8603	-1.4376	-8.9961	-17.8276	-28.2325	
8.7	36.0671	24.5836	14.6707	6.2923	-1.0869	-8.6613	-17.4774	-27.8664	
8.8	36.9214	25.2961	15.2415	6.7327	-0.7299	-8.3236	-17.1249	-27.4975	
8.9	37.7856	26.0186	15.8221	7.1818	-0.3678	-7.9832		-27.1258	
9.0	38.6598	26.7510	16.4127	7.6396	0.0000	-7.6398	-16.4127	-26.7513	-38.6598

Table 11 - Eigenvalues of the θ Equation for the Even Dirichlet and Odd Neumann Problem for $k^2 = 0.1$ and $k^2 = 0.9$

$k^2 = 0.1$		$k^2 = 0.9$	
ν	μ	ν	μ
0.0	0.2372	0	0.0928
0.1	0.2318	0.1	0.0557
0.2	0.2253	0.2	-0.0092
0.3	0.2179	0.3	-0.0480
0.4	0.2094	0.4	-0.1173
0.5	0.2000	0.5	-0.2000
0.6	0.1895	0.6	-0.2976
0.7	0.1780	0.7	-0.4112
0.8	0.1653	0.8	-0.5417
0.9	0.1516	0.9	-0.6895
1.0	0.1369	1.0	-0.8552

Table 12 - Eigenvalues of the φ Equation for the Even Dirichlet and Neumann Problems for $k'^2 = 0.1$ and $k'^2 = 0.9$

$k'^2 = 0.1$		$k'^2 = 0.9$	
ν	μ	ν	μ
0.0	0.0000	0.0	0.0000
0.1	0.0054	0.1	0.0369
0.2	0.0119	0.2	0.0825
0.3	0.0193	0.3	0.1377
0.4	0.0277	0.4	0.2038
0.5	0.0372	0.5	0.2821
0.6	0.0477	0.6	0.3738
0.7	0.0592	0.7	0.4803
0.8	0.0717	0.8	0.6029
0.9	0.0853	0.9	0.7425
1.0	0.1000	1.0	0.9000

Table 13 - Eigenvalues of the φ Equation for the Odd Dirichlet and Neumann Problems for $k'^2 = 0.1$ and $k'^2 = 0.9$

$k'^2 = 0.1$		$k'^2 = 0.9$	
ν	μ	ν	μ
0.0	-0.9490	0.0	-0.3712
0.1	-0.9408	0.1	-0.3114
0.2	-0.9312	0.2	-0.2400
0.3	-0.9200	0.3	-0.1565
0.4	-0.9074	0.4	-0.0607
0.5	-0.8932	0.5	0.0480
0.6	-0.8776	0.6	0.1700
0.7	-0.8605	0.7	0.3057
0.8	-0.8418	0.8	0.4557
0.9	-0.8217	0.9	0.6203
1.0	-0.8000	1.0	0.8000

APPENDIX C

DETERMINATION OF THE EIGENFUNCTIONS

The recurrence relations for the θ solutions of the even Dirichlet and odd Neumann problems are the same. The relationship is given by an infinite set of three term equations, the set extending toward infinity in both directions. One way of presenting this set of equations is

$$\cdots\cdots$$

$$a_{-2}A_{-3}+b_{-2}A_{-2}+c_{-2}A_{-1} = 0 \tag{C1a}$$

$$a_{-1}A_{-2}+b_{-1}A_{-1}+ c_{-1}A_0 = 0 \tag{C1b}$$

$$a_0 A_{-1}+ b_0 A_0 + c_0 A_1 = 0 \tag{C1c}$$

$$a_1 A_0 + b_1 A_1 + c_1 A_2= 0 \tag{C1d}$$

$$a_2 A_1 + b_2 A_2 + c_2 A_3 = 0 \tag{C1e}$$

$$\cdots\cdots$$

Since there is no starting point, the evaluation of the coefficient is not simple. However, a continued fraction expression similar to the ones used for the eigenvalues can be determined. For example, equation (C1d) can be written

$$\frac{A_0}{A_1} = - \frac{b_1}{a_1} - \frac{c_1/a_1}{A_1/A_2} \tag{C2}$$

but, from equation (C1e), A_1/A_2 can be written,

$$\frac{A_1}{A_2} = -\frac{b_2}{a_2} - \frac{c_2/a_2}{A_2/A_3} \tag{C3}$$

This can be repeated as many times as desired; however, it is already evident that the infinite continued fraction will have the form

$$\frac{A_0}{A_1} = -\frac{b_1}{a_1} + \frac{c_1/a_1}{b_2/a_2 +} \; \frac{c_2/a_2}{b_3/a_3 +} \; \frac{c_3/a_3}{b_4/a_4 +} \; \frac{c_4/a_4}{b_5/a_5 +} \cdots\cdots \tag{C4}$$

It is also possible to start with equation (C1c) and go in the other direction to determine that

$$\frac{A_1}{A_0} = -\frac{b_0}{c_0} + \frac{a_0/c_0}{b_{-1}/c_{-1}} \; \frac{a_{-1}/c_{-1}}{+ b_{-2}/c_{-2}} \; \frac{a_{-2}/c_{-2}}{+ b_{-3}/c_{-3}}$$

$$\frac{a_{-3}/c_{-3}}{+ b_{-4}/c_{-4}} + \cdots\cdots \tag{C5}$$

By equating (C4) and the inverse of (C5) a relationship between the eigenvalues ν and μ can be obtained. This is in essence what was done in order to determine the eigenvalues. The eigenvalue equations are actually consistency equations, since they must be satisfied in order to have a consistent relationship between the coefficients of the eigenfunctions.

The coefficients a_m, b_m, and c_m are functions of ν, μ, and σ. Since the eigenvalues have been determined already, a_m, b_m, and c_m are known quantities and it is only necessary to calculate the value of the continued fraction in equation (C4) to determine A_0/A_1. For convenience A_0 can be set equal to unity, so that A_1 is known; next A_1/A_2 can be determined, then A_2 is known. This can be repeated for as many coefficients as are desired. In order to compute A_{-1}, one could use equation (C1c) to write

$$\frac{A_{-1}}{A_0} = -\frac{b_0}{a_0} - \frac{c_0/a_0}{A_0/A_1} \tag{C6}$$

and use the information just obtained from the computation of the coefficients with positive subscripts, or one could use equation (C1b) and generate a system of equations like equation (C5). In either case, the common factor is A_0. It was found that by starting with A_0 and working toward coefficients with large magnitude subscripts, a cumulative error was generated. This occurs because the coefficients with small subscripts are generally larger than the coefficients with large subscripts; thus a small percentage error in A_1 might be very large relative to A_5. For this reason it is better to start with A_n, n large, and work back to A_0. This is the way the eigenfunction coefficients were computed in this paper.

The series converge rather quickly, so it was decided that approximately 20 terms should be more than sufficient. Initially A_{11} is assumed to be unity. Then A_{10}/A_{11} is determined and thus A_{10} is known. This is repeated for A_9/A_{10}, A_8/A_9, etc. until A_0 is determined. Next A_{-11} is assumed to be unity. A_{-10}/A_{-11} is determined and thus A_{-10} is known. This process is repeated until A_0 is determined again. The A_0 found by starting at A_{-11} is set equal to the A_0 calculated the first time and all of the negative subscripted coefficients are scaled accordingly. Then using all of the coefficients from A_{-11} to A_{11}, the eigenfunction is normalized so that it has unit magnitude or unit slope at zero argument according to whether it is even or odd respectively. The normalized coefficients which are greater than 5×10^{-4} are tabulated in the tables in Chapter III.

The continued fractions used in the calculations described in the previous paragraphs are

$$\frac{A_{m-1}}{A_m} = -\frac{b_m}{a_m} + \frac{c_m/a_m}{b_{m+1}/a_{m+1}} + \frac{c_{m+1}/a_{m+1}}{b_{m+2}/a_{m+2}} + \cdots \qquad (C7)$$

where

$$a_m = \frac{k^2}{4}\left(\frac{(4m-3)(4m-5)}{4} - \nu(\nu+1)\right) \qquad (C8a)$$

$$b_m = \left(\frac{(4m-1)^2}{4}\left(\frac{k^2}{2} - 1\right) + \frac{\nu(\nu+1)\,k^2}{2} + \mu\right) \qquad (C8b)$$

$$c_m = \frac{k^2}{4}\left(\frac{(4m+1)(4m+3)}{4} - \nu(\nu+1)\right) \qquad (C8c)$$

Equations (C7) and (C8) are just the set of equations (3.25) written in a form suitable for calculating the coefficients. Although these equations are good for all m, they are used only for $m \geq 1$, i.e., they are used to calculate only the coefficients with positive subscripts. For the coefficients with negative subscripts, the set of equations (3.25) are written in a different form.

$$\frac{A_{m+1}}{A_m} = -\frac{b_m}{c_m} + \frac{a_m/c_m}{b_{m-1}/c_{m-1} + \frac{a_{m-1}/c_{m-1}}{b_{m-2}/c_{m-2}}}$$

$$+ \frac{a_{m-2}/c_{m-2}}{b_{m-3}/c_{m-3}} + \ldots \quad \text{(C9)}$$

where a_m, b_m, and c_m are the same as before and the equation is used for $m \leq -1$.

Equations (C7), (C8), and (C9) are used to calculate the coefficients of all of the eigenfunctions of the even Dirichlet and odd Neumann problems, except those for which ν is a half integer. For these cases, either a_m or c_m is zero for some value of m, and either equation (C7) or (C9) becomes singular. For $\nu = 2n + 1/2$, n an integer, it can be shown that $A_{-n-1} = 0$, which implies that all of the coefficients with m less than -n-1 are zero. For $\nu = 2n - 1/2$, it can be shown that $A_{n+1} = 0$, which implies that all of the coefficients with m greater than n + 1 are zero. For each of these cases, one of the infinite continued fractions becomes a finite continued fraction. For example, for $\nu = 7.5$, $A_m = 0$ for $m > 4$. Thus it is assumed that $A_4 = 1$ and equation (C7) is used to find A_3, A_2, A_1, and A_0. The rest of the calculation is the same as before.

The recurrence relations for the ϕ eigenfunctions of the even Dirichlet problem and the odd Neumann problem are not the same, but the method of determining the coefficients is. For both of these problems, the eigenfunctions are represented by infinite series, but unlike the previous case the summation begins at m = 0. There are four different eigenfunction types. As an example for this discussion, consider

$$\Phi_{e1}(\phi) = \sum_{m=0}^{\infty} B_{2m+1} \cos (2m+1) \phi \tag{C10}$$

From equation (3.31) the recurrence relations are

$$B_1 \left(\left(\frac{k'^2}{2} - 1\right) + \frac{\nu(\nu+1)\, k'^2}{2} - \mu\right) + B_3 \frac{k'^2}{4} (6 - \nu(\nu+1)) = 0 \tag{C11a}$$

$$B_1 \frac{k'^2}{4} (2 - \nu(\nu+1)) + B_3 \left(9 \left(\frac{k'^2}{2} - 1\right) + \frac{\nu(\nu+1)\, k'^2}{2} - \mu\right)$$

$$+ B_5 \frac{k'^2}{4} (20 - \nu(\nu+1)) = 0 \tag{C11b}$$

$$B_{2m-1} \frac{k'^2}{4} ((2m-1)(2m) - \nu(\nu+1)) + B_{2m+1} \left((2m+1)^2 \left(\frac{k'^2}{2} - 1\right)\right.$$

$$\left. + \frac{\nu(\nu+1)\, k'^2}{2} - \mu\right) + B_{2m+3} \frac{k'^2}{4} ((2m+3)(2m+2) - \nu(\nu+1)) = 0 \tag{C11c}$$

The easiest way to solve this set of equations is to set B_1 equal to unity and solve equation (C11a) for B_3. Then equation (C11b) can be solved for B_5, etc. As was the case with the θ eigenfunctions, starting with a large coefficient and working toward the smaller ones generates a cumulative error. By using all of the recurrence equations except the first one, an infinite continued fraction can be generated. For example, if equations (C11) are written in the form

$$b_1 B_1 + c_1 B_3 = 0 \tag{C12a}$$

$$a_3 B_1 + b_3 B_3 + c_3 B_5 = 0 \tag{C12b}$$

$$a_5 B_3 + b_5 B_5 + c_5 B_7 = 0 \tag{C12c}$$

$$\cdots\cdots$$

the equation for B_1/B_3 can be written

$$\frac{B_1}{B_3} = - \frac{b_3}{a_3} + \frac{c_3/a_3}{b_5/a_5 +} \; \frac{c_5/a_5}{b_7/a_7 +} \; \frac{c_7/a_7}{b_9/a_9 +} \cdots \qquad \text{(C13)}$$

and similarly for B_3/B_5, B_5/B_7, etc.

Note that equation (C12) can be written

$$\frac{B_1}{B_3} = - \frac{c_1}{b_1} \qquad \text{(C14)}$$

Equation (C14) can be equated with equation (C13) to obtain the eigenvalue, or consistency, equation. The eigenvalue equation, equation (3.33), was actually derived by using standard procedures on the coefficient determinant. Manipulating the determinant and equating (C13) and (C14) are essentially two different ways of handling the same mathematical problem; the final result using either approach is the same. The determinantal method is more straight-forward and systemmatic, whereas the method described in this appendix lends some insight into why the eigenvalue equation must be satisfied.

From equations (C11), the continued fraction for the coefficients is

$$\frac{B_{2m-1}}{B_{2m+1}} = - \frac{b_{2m+1}}{a_{2m+1}} + \frac{c_{2m+1}/a_{2m+1}}{b_{2m+3}/a_{2m+3}}$$

$$+ \frac{c_{2m+3}/a_{2m+3}}{b_{2m+5}/a_{2m+5} +} \; \frac{c_{2m+5}/a_{2m+5}}{b_{2m+7}/a_{2m+7} +} \cdots$$

$$m \geq 1 \qquad \text{(C15)}$$

where

$$a_{2m+1} = \frac{k'^2}{4} \; ((2m-1)(2m) - \nu(\nu+1)) \qquad \text{(C16a)}$$

$$b_{2m+1} = ((2m+1)^2 \; (\frac{k'^2}{2} -1) + \frac{\nu(\nu+1) \; k'^2}{2} - \mu) \qquad \text{(C16b)}$$

$$c_{2m+1} = \frac{k'^2}{4} \; ((2m+3)(2m+2) - \nu(\nu+1)) \qquad \text{(C16c)}$$

To solve for the coefficients, B_{41} is set equal to unity, B_{39}/B_{41} is calculated, and thus B_{39} is known. This is repeated for B_{37}/B_{39}, B_{35}/B_{37}, etc., until B_1 is known. Then using all of the coefficients, B_1 through B_{41}, the eigenfunction is normalized. Tabulated values appear in Chapter III.

Convergence properties of the eigenfunction expansion can be determined by considering the continued fraction expression for the ratio of two successive coefficients, such as A_n/A_{n-1}, as n approaches infinity. The subscript n can be m or 2m, depending on the type of eigenfunction. Ince has investigated this and determined that

$$\lim_{n \to \infty} \frac{A_n}{A_{n-1}} < k^2 \quad .$$

He concludes that the series do converge, and that the convergence is more rapid than a power series expansion. For a more complete discussion of this subject, see Ince[10,11].

When ν is an integer, the above procedure for determining the eigenfunction coefficients breaks down, because either a_{2m+1} or

c_{2m+1} is zero for some m. For integral values of ν, the eigenfunctions are Lamé polynomials and the series are no longer infinite. All of the θ and ϕ eigenfunctions of the odd Dirichlet and even Neumann problem fall into this category. Although these eigenfunctions are written in eight different forms (see equations (3.35)-(3.38) and (3.51)-(3.54)) with eight different sets of recurrence relations, the method of determining the coefficients is the same. A typical recurrence relation is given by equation (3.42), which appears below

$$A_{2m-2}\,\frac{k^2}{4}\left((2m)(2m-1) - \nu(\nu+1)\right)$$

$$+ A_{2m}\left((2m)^2\left(\frac{k^2}{2} - 1\right) + \frac{\nu(\nu+1)\,k^2}{2} + \mu\right)$$

$$+ A_{2m+2}\,\frac{k^2}{4}\left((2m)(2m+1) - \nu(\nu+1)\right) = 0$$

$$A_0 = 0 \qquad m \geq 1 \tag{C17}$$

This set of equations is very similar to the set of equations (C11). The first equation contains two coefficients, A_2 and A_4; the second equation contains A_2, A_4, and A_6; and the rest follow a three term recursive relationship. The main difference between this problem and the previous problem is that there are only a finite number of coefficients. When $\nu = 9$, the largest value considered here, the maximum number of coefficients is four; A_2, A_4, A_6, and A_8. These coefficients are found simply by solving each equation one at a time starting with the first. With a simplified notation, the set of equations (C17) can be written (for $\nu = 9$)

$$b_2 A_2 + c_2 A_4 = 0 \tag{C18a}$$

$$a_4 A_2 + b_4 A_4 + c_4 A_6 = 0 \tag{C18b}$$

$$a_6 A_4 + b_6 A_6 + c_6 A_8 = 0 \tag{C18c}$$

with the a_m's, b_m's, and c_m's easily identifiable from equation (C17). With

$$A_2 = 1$$

$$A_4 = -\frac{b_2}{c_2} \tag{C19a}$$

$$A_6 = -\frac{a_4}{c_4} - \frac{b_4}{c_4} A_4 \tag{C19b}$$

$$A_8 = -\frac{a_6}{c_6} A_4 - \frac{b_6}{c_6} A_6 \tag{C19c}$$

The coefficients are then normalized and tabulated. The fact that the eigenfunction has a finite number of terms is indicated in the tables by the statement $A_m = 0$, $m > 8$. There are seven other sets of recurrence relations for the odd Dirichlet and even Neumann problem. They all contain a finite number of terms, they all have integer values of ν for eigenvalues, and they are all evaluated using this technique.

The same problem concerning accuracy exists here as before, i.e., a cumulative error is generated. The problem could be cast in the form of a continued fraction, but it was felt that this would unduly complicate it. The number of coefficients involved is small enough that the accumulated error is usually insignificant. However, it is recommended for the sake of accuracy that the continued fraction method be used if eigenfunctions are computed for $\nu > 9$.

Arscott has tabulated the Lamé polynomials for the values of ν from one to thirty, and for $k^2 = 0.1, 0.2, \ldots\ldots, 0.9$. They are not in the same form as the Lamé polynomials calculated here, but they can be converted with the help of some trigonometric identities. Several of his polynomials were converted and compared with the functions computed in this paper. The comparison indicated good accuracy for most cases, however it did show some evidence of the cumulative error problem. For $\nu = 9$, the eigenfunction contains four or five terms. For some of these eigenfunctions, the magnitude of the coefficients is decreasing as the subscript is increasing. For these eigenfunctions, the last coefficient usually lacks precision; however, this is relatively unimportant because it is always small. This lack of precision is a consequence of subtracting two large numbers which are nearly equal.

APPENDIX D

VARIATIONAL METHOD FOR DETERMINING EIGENVALUES AND EIGENFUNCTIONS

It was shown in Appendix A that

$$\frac{\langle y, L y \rangle}{\langle y, \rho y \rangle} = \lambda \tag{D1}$$

where L is the two-dimensional Sturm-Liouville operator defined by equation (A2), ρ is the weight function defined by equation (A3), λ is an eigenvalue, and y is the corresponding eigenfunction. It was pointed out in Appendix A that the pertinent eigenvalues are real and positive or zero. The method employed here is essentially the Rayleigh-Ritz method with some modifications due to the two-dimensional nature of the problem and due to computational difficulties. Thus

$$\frac{\langle w, L w \rangle}{\langle w, \rho w \rangle} \geq \lambda_1 \tag{D2}$$

where w is any function in the domain of L, and λ_1 is the lowest eigenvalue. Equation (D2) is the basis for a method of determining the eigenvalues and eigenfunctions.

Consider the following eigenfunctions of the odd Neumann problem:

$$\begin{aligned}\Theta(\theta) = &- A_0 \sin \theta/2 + A_1 \sin 3\theta/2 - A_{-1} \sin 5\theta/2 \\ &+ A_2 \sin 7\theta/2 - A_{-2} \sin 9\theta/2 + \dots\dots\end{aligned} \tag{D3}$$

$$\Phi(\phi) = B_1 \sin \phi + B_3 \sin 3\phi + B_5 \sin 5\phi + \ldots \tag{D4}$$

$$w(\theta, \phi) = \Theta(\theta)\, \Phi(\phi) \tag{D5}$$

As a first approximation, let

$$B_1 = A_0 = 1 \tag{D6a}$$

$$A_{-1} = A_2 = A_{-2} = B_3 = B_5 = 0 \tag{D6b}$$

Thus,

$$w(\theta, \phi) = (-\sin \theta/2 + A_1 \sin 3\theta/2) \sin \phi \tag{D7}$$

Insert this into equation (D2) and evaluate the two-dimensional integrals. The result is

$$\frac{(A_1)^2\, T + A_1\, U + V}{(A_1)^2\, X + A_1\, Y + Z} \geq \lambda_1 \tag{D8}$$

where T, U, V, X, Y, and Z follow from the integral evaluation. Minimizing the left hand side leads to a quadratic equation in A_1. A basic assumption in this method is that $A_0 = 1$ is the largest coefficient. Thus if either or both roots for A_1 are larger than one, they are discarded. If both are less than one, the smallest is used. Inserting this value of A_1 into equation (D8), the first approximation for λ_1 is found.

Next, assume that

$$w(\theta, \phi) = (-\sin \theta/2 + A_1 \sin 3\theta/2 - A_{-1} \sin 5\theta/2) \sin \phi \tag{D9}$$

The procedure is repeated for the new unknown, A_{-1}, and a second approximation for λ_1 is found. This procedure is repeated until all of the coefficients A_1, A_{-1}, A_2, A_{-2}, B_3, and B_5 are known.

Next the process is repeated again for small changes in the coefficients, i.e., $w\ (\theta, \phi)$ is assumed to have the form

$$w\ (\theta, \phi) = (-\sin\theta/2 + (A_1 + \Delta A_1)\sin 3\theta/2 - A_{-1}\sin 5\theta/2 + A_2 \sin 7\theta/2 - A_{-2}\sin 9\theta/2) \cdot (\sin\phi + B_3\sin 3\phi + B_5 \sin 5\phi) \tag{D10}$$

where ΔA_1 is the unknown. This first order correction is determined using the same procedure as before. It is repeated for ΔA_{-1}, ΔA_2, ΔA_{-2}, ΔB_3, and ΔB_5. The process could be repeated again for a second order correction if desired. This was not done for the eigenvalues and coefficients of the eigenfunctions given in Table 14.

The computation of other eigenvalues is similar. The first eigenvalue was found by assuming that A_0 and B_1 are the largest coefficients. Other eigenvalues are found by assuming that other coefficients are maximum; for example, the second eigenvalue in Table 14 results from assuming A_1 and B_1 to be maximum.

The main difficulty with this method is that it is based on assumptions which, in turn, are mostly based on intuition. Thus there is no assurance that all of the eigenvalues will be found, or even if all of the numbers are actually eigenvalues. The method works quite well for the lowest eigenvalues, however, as can be seen in Table 14.

The coefficients in this table were normalized so that the maximum values for both sets of coefficients agree.

Table 14 - Comparison of Eigenvalues and Eigenfunctions Determined by the Variational Method and the Continued Fraction Method

	Variational	Fraction
ν	0.815	0.814
A_0	-1.473	-1.473
A_1	0.215	0.214
A_{-1}	0.030	0.030
A_2	0.007	0.007
A_{-2}	0.001	0.002
B_1	0.964	0.964
B_3	0.011	0.010
B_5	0.001	0.001
ν	1.597	1.595
A_0	0.183	0.174
A_1	0.689	0.689
A_{-1}	-0.028	-0.026
A_2	-0.005	-0.005
A_{-2}	-0.001	-0.001
B_1	1.160	1.160
B_3	-0.049	-0.048
B_5	-0.002	-0.003
ν	1.997	1.955
A_0	-1.360	-1.360
A_1	0.581	0.568
A_{-1}	0.186	0.179
A_2	-0.018	-0.016
A_{-2}	0.004	0.004
B_2	0.496	0.496
B_4	0.001	0.002
ν	2.806	2.803
A_0	0.414	0.346
A_1	0.718	0.718
A_{-1}	-0.209	-0 180
A_2	-0.092	-0.091
A_{-2}	0.004	0.003
B_2	0.583	0.583
B_4	-0.038	-0.038
ν	2.984	2.990
A_0	-1.720	-1.720
A_1	1.032	1.025
A_{-1}	0.427	0.429
A_2	-0.105	-0.104
A_{-2}	-0.010	-0.011
B_1	0	0.032
B_3	0.322	0.322

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13. ABSTRACT

In this report, the exact solution for the electromagnetic field diffracted by a perfectly conducting plane angular sector is determined. The problem is a three-dimensional vector problem and the solution is presented in the form of a dyadic Green's function, which is the most general form of solution possible. Thus the vector fields as well as the current on the sector may be determined for any given source excitation.

The corner angle of the plane angular sector is arbitrary, varying between zero and π. Special cases of the plane angular sector are the half plane and the quarter plane. To find the fields for larger angles, such as the corner of an aperture, Babinet's principle can be used.

The dyadic Green's function is composed of vector wave functions, which in turn are composed of scalar wave functions. The problem is solved in a sphero-conal coordinate system. In this system, the plane angular sector is one of the coordinate surfaces, so the separation of variables technique is used to find the scalar wave functions. They consist of spherical Bessel's functions and Lamé functions. The Lamé functions are solutions of two coupled differential equations. These equations are solved for the special case of a quarter plane scatterer. The first 192 eigenvalues and eigenfunctions are computed and tabulated.

The fields and currents close to the tip of the quarter plane are presented. These fields and currents have been the subject of much conjecture by several authors. It is shown that the dominant field behaves as $r^{\nu-1}$, where the lowest value of the eigenvalue ν is 0.296. The far fields for infinitesimal dipole sources very close to the tip are also determined and several patterns are presented.

As with any exact solution of a complex problem, the results are not simple. It is felt, however, that the exact solution obtained here lays the foundation for subsequent work. For instance, it should be possible to use this work to determine an asymptotic approximation and thus derive a "diffraction coefficient" for the tip. This would be very useful in Keller's "Geometrical Theory of Diffraction." Without any additional work, there are sufficient numerical results presented in this report to determine the fields within approximately one wavelength of the tip for any source, or the fields everywhere for a source within one wavelength of the tip.

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